

# Some new results on higher energies \*

Shkredov I.D.

Annotation.

*In the paper we develop the method of higher energies. New upper bounds for the additive energies of convex sets, sets  $A$  with small  $|AA|$  and  $|A(A+1)|$  are obtained. We prove new structural results, including higher sumsets, and develop the notion of dual popular difference sets.*

## 1 Introduction

The method of higher energies (or, in other words, the method of higher moments of convolutions) was introduced in [20], was developed in [22, 27] and found a series of applications in [10, 14, 15, 16, 21, 25, 26]. In the paper we obtain some new results in the direction, using so-called operator (or eigenvalues) method from [24, 25], which we recall in section 4.

Our main results are contained in three sections 5–7. In section 5 we apply the eigenvalues method to obtain new upper bounds for the additive energy of some families of sets. Let us formulate just one result in the direction which concerns convex subset (that is the image of a convex map) of  $\mathbb{R}$ .

**Theorem 1** *Let  $A \subseteq \mathbb{R}$  be a convex set. Then*

$$E(A) \ll |A|^{\frac{32}{13}} \log^{\frac{71}{65}} |A|. \quad (1)$$

Here  $E(A)$ , so-called, the *additive energy* of our set  $A$ , which equals the number of solution of the equation  $a_1 - a_2 = a_3 - a_4$ , where  $a_1, a_2, a_3, a_4 \in A$ . Constant  $\frac{5}{2}$  instead of  $\frac{32}{13}$  was obtained in [12], and after that this constant was improved to  $\frac{89}{36}$  in [25], using the eigenvalues method again.

The next section contains so-called structural results. The most important example of such statements in additive combinatorics is, of course, beautiful Freiman's theorem on sets with small doubling, or, in other words, sets having small sumset, which gives a full description of the family of sets (see [28]). In the paper by structural results we mean another thing, namely,

---

\*This work was supported by grant RFFI NN 06-01-00383, 11-01-00759, Russian Government project 11.G34.31.0053, Federal Program "Scientific and scientific-pedagogical staff of innovative Russia" 2009–2013 and grant Leading Scientific Schools N 2519.2012.1.

having some condition on a set (basically, on higher energies of this set) we wish to find some subsets of the set having small doubling or large additive energy. As an example of such type of results, we recall a strong structural theorem from [2].

**Theorem 2** *Let  $A \subseteq \mathbf{G}$  be a symmetric set,  $\tau_0, \sigma_0$  be nonnegative real numbers and  $A$  has the property that for any  $A_* \subseteq A$ ,  $|A_*| \gg |A|$  the following holds  $E(A_*) \gg E(A) = |A|^{2+\tau_0}$ . Suppose that  $T_4(A) \ll |A|^{4+3\tau_0+\sigma_0}$ . Then there exists a function  $f_{\tau_0} : (0, 1) \rightarrow (0, \infty)$  with  $f_{\tau_0}(\eta) \rightarrow 0$  as  $\eta \rightarrow 0$  and a number  $\alpha \geq 0$  such that there are sets  $X_j, H_j \subseteq \mathbf{G}$ ,  $B_j \subseteq A$ ,  $j \in [|A|^{\alpha-f_{\tau_0}(\sigma_0)}]$  with*

$$|H_j| \ll |A|^{\tau_0+\alpha+f_{\tau_0}(\sigma_0)}, \quad |X_j| \ll |A|^{1-\tau_0-2\alpha+f_{\tau_0}(\sigma_0)},$$

$$|H_j - H_j| \ll |H_j|^{1+f_{\tau_0}(\sigma_0)},$$

$$|(X_j + H_j) \cap B_j| \gg |A|^{1-\alpha-f_{\tau_0}(\sigma_0)},$$

and  $B_i \cap B_j = \emptyset$  for all  $i \neq j$ .

Here  $T_4(A)$  is the number solution of the equation  $a_1 + a_2 + a_3 + a_4 = a'_1 + a'_2 + a'_3 + a'_4$ ,  $a_1, a_2, a_3, a_4, a'_1, a'_2, a'_3, a'_4 \in A$ .

In the paper we need in a generalization of the notion of the additive energy of a set. Namely, for all real  $s \geq 1$  put

$$E_s(A) = \sum_x |A \cap (A - x)|^s, \quad (2)$$

Quantities  $E_s(A)$  from (2) are exactly that we call higher energies.

Let us formulate our two main structural results. The weaker forms of the first one were proved in [22] and [25]. From some point of view these type of statements can be called an optimal version of Balog–Szemerédi–Gowers theorem, see [22].

**Theorem 3** *Let  $A \subseteq \mathbf{G}$  be a set,  $E(A) = |A|^3/K$ , and  $E_3(A) = M|A|^4/K^2$ . Then there is a set  $A' \subseteq A$  such that*

$$|A'| \gg M^{-10} \log^{-15} M \cdot |A|, \quad (3)$$

and

$$|nA' - mA'| \ll (M^9 \log^{14} M)^{6(n+m)} K |A'| \quad (4)$$

for every  $n, m \in \mathbb{N}$ .

Interestingly, that the generality of Theorem 3 allows us to prove a "non-trivial" version of Theorem 1, that is a bound of the form  $E(A) \ll |A|^{5/2-\varepsilon_0}$ , where  $\varepsilon_0 > 0$  is an absolute constant. A similar proof takes place in the case of multiplicative subgroups of  $\mathbb{Z}/p\mathbb{Z}$ ,  $p$  is a prime number (see Remark 36).

The second structural result is the following.

**Theorem 4** *Let  $A \subseteq \mathbf{G}$  be a set,  $E_{3/2}(A) = |A|^{5/2}/K^{1/2}$ , and  $T_4(A) = M|A|^7/K^3$ . Then there is a set  $A' \subseteq A$  such that*

$$|A'| \gg \frac{|A|}{MK}, \quad (5)$$

and

$$E(A') \gg \frac{|A'|^3}{M}. \quad (6)$$

It is easy to see that Theorem 4 is tight (see e.g. Remark 39). The assumption of our result is stronger than the assumption of Theorem 2.

Popular difference sets are very simple and important objects in additive combinatorics (see e.g. [8, 9, 28]). In section 7 we develop an idea of Bateman–Katz from [1, 2] that every popular set has a companion, which we call a dual popular set. As an application, our method allows find a nontrivial relation between  $E(A)$  and  $E_s(A)$ ,  $s \in [1, 2]$ , see Theorem 43 or Corollary 46. It is interesting that for  $s > 2$  there is no such connection at all.

Note, finally, that the arguments of the paper are elementary in the sense that they do not use Fourier transform.

The author is grateful to Tomasz Schoen, Sergey Konyagin, and Misha Rudnev for fruitful discussions and explanations.

## 2 Definitions

Let  $\mathbf{G}$  be an abelian group. If  $\mathbf{G}$  is finite then denote by  $N$  the cardinality of  $\mathbf{G}$ . We define two types of convolutions on  $\mathbf{G}$

$$(f * g)(x) := \sum_{y \in \mathbf{G}} f(y)g(x - y) \quad \text{and} \quad (f \circ g)(x) := \sum_{y \in \mathbf{G}} f(y)g(y + x) = (f * g^c)(-x),$$

where for a function  $f : \mathbf{G} \rightarrow \mathbb{C}$  we put  $f^c(x) := f(-x)$ . Clearly,  $(f * g)(x) = (g * f)(x)$  and  $(f \circ g)(x) = (g \circ f)(-x)$ ,  $x \in \mathbf{G}$ . The  $k$ -fold convolution,  $k \in \mathbb{N}$  we denote by  $*_k$ , so  $*_k := (*_{k-1})$ .

We use in the paper the same letter to denote a set  $S \subseteq \mathbf{G}$  and its characteristic function  $S : \mathbf{G} \rightarrow \{0, 1\}$ . Write  $E(A, B)$  for the *additive energy* of two sets  $A, B \subseteq \mathbf{G}$  (see e.g. [28]), that is

$$E(A, B) = |\{a_1 + b_1 = a_2 + b_2 : a_1, a_2 \in A, b_1, b_2 \in B\}|.$$

If  $A = B$  we simply write  $E(A)$  instead of  $E(A, A)$ . Clearly,

$$E(A, B) = \sum_x (A * B)(x)^2 = \sum_x (A \circ B)(x)^2 = \sum_x (A \circ A)(x)(B \circ B)(x). \quad (7)$$

Let

$$T_k(A) := \sum_x (A *_k A)^2(x) = |\{a_1 + \dots + a_k = a'_1 + \dots + a'_k : a_1, \dots, a_k, a'_1, \dots, a'_k \in A\}|.$$

Let also

$$\sigma_k(A) := (A *_{k-1} A)(0) = |\{a_1 + \dots + a_k = 0 : a_1, \dots, a_k \in A\}|.$$

Notice that for a symmetric set  $A$  that is  $A = -A$  one has  $\sigma_2(A) = |A|$  and  $\sigma_{2k}(A) = \mathsf{T}_k(A)$ . If  $\psi : \mathbf{G} \rightarrow \mathbb{C}$  is a function then we write

$$\sigma_\psi(A) = \sigma(\psi, A) := \sum_x \psi(x)(A \circ A)(x).$$

So, if  $P \subseteq \mathbf{G}$  is another set then put  $\sigma_P(A) := \sum_{x \in P} (A \circ A)(x)$ . Similarly, write  $\mathsf{E}_P(A) := \sum_{x \in P} (A \circ A)^2(x)$ .

For a sequence  $s = (s_1, \dots, s_{k-1})$  put  $A_s^B = B \cap (A - s_1) \dots \cap (A - s_{k-1})$ . If  $B = A$  then write  $A_s$  for  $A_s^A$ . Let

$$\mathsf{E}_k(A) = \sum_{x \in \mathbf{G}} (A \circ A)(x)^k = \sum_{s_1, \dots, s_{k-1} \in \mathbf{G}} |A_s|^2 \quad (8)$$

and

$$\mathsf{E}_k(A, B) = \sum_{x \in \mathbf{G}} (A \circ A)(x)(B \circ B)(x)^{k-1} = \sum_{s_1, \dots, s_{k-1} \in \mathbf{G}} |B_s^A|^2 \quad (9)$$

be the higher energies of  $A$  and  $B$ . The second formulas in (8), (9) can be considered as the definitions of  $\mathsf{E}_k(A)$ ,  $\mathsf{E}_k(A, B)$  for non integer  $k$ ,  $k \geq 1$ . As above for a set  $P \subseteq \mathbf{G}$  write  $\mathsf{E}_k^P(A) := \sum_{s \in P} |A_s|^k$  and for a set  $\mathcal{P} \subseteq \mathbf{G}^{k-1}$  put

$$\mathsf{E}_k^{\mathcal{P}}(A) := \sum_{(s_1, \dots, s_{k-1}) \in \mathcal{P}} |A_s|^2.$$

Clearly,

$$\begin{aligned} \mathsf{E}_{k+1}(A, B) &= \sum_x (A \circ A)(x)(B \circ B)(x)^k \\ &= \sum_{x_1, \dots, x_{k-1}} \left( \sum_y A(y)B(y+x_1) \dots B(y+x_k) \right)^2 = \mathsf{E}(\Delta_k(A), B^k), \end{aligned} \quad (10)$$

where

$$\Delta(A) = \Delta_k(A) := \{(a, a, \dots, a) \in A^k\}.$$

We also put  $\Delta(x) = \Delta(\{x\})$ ,  $x \in \mathbf{G}$ .

Quantities  $\mathsf{E}_k(A, B)$  can be written in terms of generalized convolutions.

**Definition 5** Let  $k \geq 2$  be a positive number, and  $f_0, \dots, f_{k-1} : \mathbf{G} \rightarrow \mathbb{C}$  be functions. Write  $F$  for the vector  $(f_0, \dots, f_{k-1})$  and  $x$  for vector  $(x_1, \dots, x_{k-1})$ . Denote by

$$\mathcal{C}_k(f_0, \dots, f_{k-1})(x_1, \dots, x_{k-1})$$

the function

$$\mathcal{C}_k(F)(x) = \mathcal{C}_k(f_0, \dots, f_{k-1})(x_1, \dots, x_{k-1}) = \sum_z f_0(z)f_1(z+x_1) \dots f_{k-1}(z+x_{k-1}).$$

Thus,  $\mathcal{C}_2(f_1, f_2)(x) = (f_1 \circ f_2)(x)$ . If  $f_1 = \dots = f_k = f$  then write  $\mathcal{C}_k(f)(x_1, \dots, x_{k-1})$  for  $\mathcal{C}_k(f_1, \dots, f_k)(x_1, \dots, x_{k-1})$ .

In particular,  $(\Delta_k(B) \circ A^k)(x_1, \dots, x_k) = \mathcal{C}_{k+1}(B, A, \dots, A)(x_1, \dots, x_k)$ ,  $k \geq 1$ .

The following lemma from [25] contains all basic properties of quantities  $\mathcal{C}_k(f_0, \dots, f_{k-1})$ .

**Lemma 6** *For any functions the following holds*

$$\begin{aligned} \sum_{x_1, \dots, x_{l-1}} \mathcal{C}_l(f_0, \dots, f_{l-1})(x_1, \dots, x_{l-1}) \mathcal{C}_l(g_0, \dots, g_{l-1})(x_1, \dots, x_{l-1}) &= \\ = \sum_z (f_0 \circ g_0)(z) \dots (f_{l-1} \circ g_{l-1})(z) &\quad \textbf{(scalar product)}, \end{aligned} \quad (11)$$

moreover

$$\begin{aligned} \sum_{x_1, \dots, x_{l-1}} \mathcal{C}_l(f_0)(x_1, \dots, x_{l-1}) \dots \mathcal{C}_l(f_{k-1})(x_1, \dots, x_{l-1}) &= \\ = \sum_{y_1, \dots, y_{k-1}} \mathcal{C}_k^l(f_0, \dots, f_{k-1})(y_1, \dots, y_{k-1}) &\quad \textbf{(multi-scalar product)}, \end{aligned} \quad (12)$$

and

$$\begin{aligned} \sum_{x_1, \dots, x_{l-1}} \mathcal{C}_l(f_0)(x_1, \dots, x_{l-1}) (\mathcal{C}_l(f_1) \circ \dots \circ \mathcal{C}_l(f_{k-1}))(x_1, \dots, x_{l-1}) &= \\ = \sum_z (f_0 \circ \dots \circ f_{k-1})^l(z) &\quad (\sigma_k \text{ for } \mathcal{C}_l). \end{aligned} \quad (13)$$

Generalizing the notion of  $\sigma_P(A)$ ,  $P \subseteq \mathbf{G}$  we define for a set  $\mathcal{P} \subseteq \mathbf{G}^{k-1}$ ,  $k \geq 2$  the quantity

$$\sigma_{\mathcal{P}}(A) := \sum_{(s_1, \dots, s_{k-1}) \in \mathcal{P}} \mathcal{C}_k(A)(s_1, \dots, s_{k-1}).$$

Let  $f_1, \dots, f_t : \mathbf{G} \rightarrow \mathbb{C}$  be functions. Tensor power of the functions is defined as

$$(f_1 \otimes f_2 \otimes \dots \otimes f_t)(x_1, \dots, x_t) = f_1(x_1) f_2(x_2) \dots f_t(x_t).$$

Tensor power of a single function  $f$  is denoted by  $(f^{\otimes})(x_1, \dots, x_t) = \prod_{j=1}^t f(x_j)$ . So, with some abuse of the notation we do not write the number  $t$  in the definition of tensor power. It is easy to see that

$$(g \circ f)^{\otimes} = (g^{\otimes} \circ f^{\otimes}) \quad \text{and} \quad (g * f)^{\otimes} = (g^{\otimes} * f^{\otimes}) \quad (14)$$

and moreover

$$\mathcal{C}_k(f_0^{\otimes}, \dots, f_{k-1}^{\otimes}) = \mathcal{C}_k^{\otimes}(f_0, \dots, f_{k-1}). \quad (15)$$

For a positive integer  $n$ , we set  $[n] = \{1, \dots, n\}$ . All logarithms used in the paper are to base 2. By  $\ll$  and  $\gg$  we denote the usual Vinogradov's symbols. We write  $\ll_M$  and  $\gg_M$  if there is a dependence on a constant  $M$ .

### 3 Preliminaries

At the beginning of the section we collect some results about matrices. We need in a lemma on singular decomposition (see e.g. [22]).

**Lemma 7** *Let  $n, m$  be two positive integers,  $n \leq m$ , and let  $X, Y$  be sets of cardinalities  $n$  and  $m$ , respectively. Let also  $\mathbf{M} = \mathbf{M}(x, y)$ ,  $x \in X$ ,  $y \in Y$  be  $n \times m$  complex (real) matrix. Then there are complex (real) functions  $u_j$  defined on  $X$ ,  $v_j$  defined on  $Y$ , and non-negative numbers  $\lambda_j$  such that*

$$\mathbf{M}(x, y) = \sum_{j=1}^n \lambda_j u_j(x) \overline{v_j(y)}, \quad (16)$$

where  $\{u_j\}$ ,  $j \in [n]$ , and  $\{v_j\}$ ,  $j \in [n]$  form two orthonormal sequences, and

$$\lambda_1 = \max_{w \neq 0} \frac{\|\mathbf{M}w\|_2}{\|w\|_2}, \quad \lambda_2 = \max_{w \neq 0, w \perp u_1} \frac{\|\mathbf{M}w\|_2}{\|w\|_2}, \dots, \lambda_n = \max_{w \neq 0, w \perp u_1, \dots, w \perp u_{n-1}} \frac{\|\mathbf{M}w\|_2}{\|w\|_2}. \quad (17)$$

Further

- $\mathbf{M}u_j = \lambda_j v_j$ ,  $j \in [n]$ .
- The numbers  $\lambda_j^2$  and the vectors  $u_j$  are all eigenvalues and eigenvectors of the matrix  $\mathbf{M}^* \mathbf{M}$ .
- The numbers  $\lambda_j^2$  and the vectors  $v_j$  form  $n$  eigenvalues and eigenvectors of the matrix  $\mathbf{M} \mathbf{M}^*$ . Another  $(m - n)$  eigenvalues of  $\mathbf{M} \mathbf{M}^*$  equal zero.
- We have  $\sum_{j=1}^n \lambda_j^2 = \sum_{x,y} |\mathbf{M}^2(x, y)|$ , and

$$\sum_{j=1}^n \lambda_j^4 = \sum_{x, x'} \left| \sum_y \mathbf{M}(x, y) \overline{\mathbf{M}(x', y)} \right|^2. \quad (18)$$

We will call functions  $\{u_j\}$ ,  $j \in [n]$ , and  $\{v_j\}$ ,  $j \in [n]$  as singular functions.

Now we recall the well-known theorem of Perron–Frobenius about the dominate eigenvalue and correspondent nonnegative eigenvector of nonnegative matrices (see e.g. [11], chapter 8). By  $\rho(M)$  denote the spectral radius of a square matrix  $M$ .

**Theorem 8** *Let  $M$  be a real square matrix with nonnegative entries. Then eigenvalue  $\rho(M)$  corresponds to a nonnegative eigenvector. Conversely, if  $M$  has a strictly positive eigenvector then this eigenvector corresponds to  $\rho(M)$ .*

Also we need in a particular convex property of eigenvalues (see e.g. [11]).

**Lemma 9** *Let  $M$  be a normal  $(n \times n)$  matrix with eigenvalues  $\mu_1, \dots, \mu_n$  and  $f$  be a convex function on  $n$  complex variables. Then*

$$\max_{x_1, \dots, x_n} f(\langle Mx_1, x_1 \rangle, \dots, \langle Mx_n, x_n \rangle) = \max_{i_1, \dots, i_n} f(\mu_{i_1}, \dots, \mu_{i_n}),$$

where the left maximum is taken over all systems of orthonormal vectors  $x_1, \dots, x_n$ , and the right maximum is taken over all permutations of  $\{1, 2, \dots, n\}$ .

Now recall some combinatorial results.

The first lemma is a special case of Lemma 2.8 from [27].

**Lemma 10** *Let  $A$  be a subset of an abelian group. Then for every  $k, l \in \mathbb{N}$*

$$\sum_{\substack{s, t: \\ \|s\|=k-1, \|t\|=l-1}} \mathbf{E}(A_s, A_t) = \mathbf{E}_{k+l}(A),$$

where  $\|x\|$  denote the number of components of vector  $x$ .

We need in the Balog–Szemerédi–Gowers theorem, see [28] section 2.5 and also [19].

**Theorem 11** *Let  $\alpha \in (0, 1]$  be a real number,  $A$  and  $B$  be finite sets of an abelian group, and  $|A| \geq |B|$ . If  $\mathbf{E}(A, B) = \alpha|A|^3$ , then there exist sets  $A' \subseteq A$  and  $B' \subseteq B$  such that  $|A'| \gg \alpha|A|$ ,  $|B'| \gg \alpha|B|$  and*

$$|A' + B'| \ll \alpha^{-5}|A|.$$

Recall that a set  $A = \{a_1, \dots, a_n\} \subseteq \mathbb{R}$  is called *convex* if  $a_i - a_{i-1} < a_{i+1} - a_i$  for every  $2 \leq i \leq n-1$ . We need in a lemma, see e.g. [21], [12] or [15].

**Lemma 12** *Let  $A$  be a convex set,  $A' \subseteq A$ , and  $B$  be an arbitrary set. Then*

$$|A' + B| \gg |A'|^{3/2}|B|^{1/2}|A|^{-1/2}. \quad (19)$$

Arranging  $(A *_{k-1} A)(x_1) \geq (A *_{k-1} A)(x_2) \geq \dots$ , we have

$$(A *_{k-1} A)(x_j) \ll_k |A|^{k-\frac{4}{3}(1-2^{-k})} j^{-\frac{1}{3}}. \quad (20)$$

In particular

$$\mathbf{E}_3(A) \ll |A|^3 \log |A|,$$

and

$$\mathbf{E}(A, B) \ll |A||B|^{\frac{3}{2}}.$$

As was realized by Li [15] (see also [22]) that subsets  $A$  of real numbers with small multiplicative doubling looks like convex sets. More precisely, the following lemma from [22] holds.

**Lemma 13** *Let  $A, B \subseteq \mathbb{R}$  be finite sets and let  $|AA| = M|A|$ . Then arranging  $(A \circ B)(x_1) \geq (A \circ B)(x_2) \geq \dots$ , we have*

$$(A \circ B)(x_j) \ll (M \log M)^{2/3} |A|^{1/3} |B|^{2/3} j^{-1/3}.$$

In particular

$$\mathbf{E}(A, B) \ll M \log M |A||B|^{3/2}.$$

## 4 Operators

In the section we describe the family of operators (finite matrices). Using such operators, we obtain a series of inequalities from papers [15], [20], [21], [25] and others by uniform way. Also we prove several lemmas which we need in the next sections. Our notations here differ from the paper [24] and do not use Fourier transform.

Let  $g : \mathbf{G} \rightarrow \mathbb{C}$  be a function, and  $A, B \subseteq \mathbf{G}$  be two finite sets. Suppose that  $|B| \leq |A|$ . By  $T_{A,B}^g$  denote the rectangular matrix

$$T_{A,B}^g(x, y) = g(x - y)A(x)B(y), \quad (21)$$

and by  $\tilde{T}_{A,B}^g(x, y)$  denote the another rectangular matrix

$$\tilde{T}_{A,B}^g(x, y) = g(x + y)A(x)B(y). \quad (22)$$

Let us describe the simplest properties of matrices  $T_{A,B}^g$  and  $\tilde{T}_{A,B}^g$ . By Lemma 7, we have

$$T_{A,B}^g(x, y) = \sum_{j=0}^{|B|-1} \lambda_j(T_{A,B}^g) u_j(x) v_j(y)$$

and similar for  $\tilde{T}_{A,B}^g$ . Here  $u_j, v_j$  are singularfunctions. We arrange the eigenvalues in order of magnitude

$$\lambda_0(T_{A,B}^g) \geq \lambda_1(T_{A,B}^g) \geq \cdots \geq \lambda_{|B|-1}(T_{A,B}^g),$$

and similar for  $\tilde{T}_{A,B}^g$ . We call  $\lambda_0$  the main eigenvalue and  $u_0, v_0$  the main singularfunctions. Clearly,

$$T_{A,B}^g(T_{A,B}^g)^*(y, y') = B(y)B(y')\mathcal{C}_3(A, g, \bar{g})(-y, -y'), \quad (23)$$

$$\tilde{T}_{A,B}^g(\tilde{T}_{A,B}^g)^*(y, y') = B(y)B(y')\mathcal{C}_3(A, g, \bar{g})(y, y'), \quad (24)$$

$$(T_{A,B}^g)^*T_{A,B}^g(x, x') = A(x)A(x')\mathcal{C}_3(B, \bar{g}^c, g^c)(-x, -x'), \quad (25)$$

$$(\tilde{T}_{A,B}^g)^*\tilde{T}_{A,B}^g(x, x') = A(x)A(x')\mathcal{C}_3(B, \bar{g}, g)(x, x'). \quad (26)$$

For real  $g$ , we get

$$(\tilde{T}_{A,B}^g)^* = \tilde{T}_{B,A}^g,$$

and for even real  $g$ , we obtain

$$(T_{A,B}^g)^* = T_{B,A}^g.$$

By Lemma 7, we have

$$\sum_{j=0}^{|B|-1} \lambda_j^2(T_{A,B}^g) = \sum_{x,y} |g(x - y)|^2 A(x)B(y) \quad \text{and} \quad \sum_{j=0}^{|B|-1} \lambda_j^2(\tilde{T}_{A,B}^g) = \sum_{x,y} |g(x + y)|^2 A(x)B(y). \quad (27)$$



Further

$$\begin{aligned} \sum_j \lambda_j^4(T_{A,B}^g) &= \sum_{y,y'} B(y)B(y') |\mathcal{C}_3(A, g, \bar{g})(-y, -y')|^2 = \\ &= \sum_{x,x'} A(x)A(x') |\mathcal{C}_3(B, \bar{g}^c, g^c)(-x, -x')|^2, \end{aligned} \quad (28)$$

and

$$\begin{aligned} \sum_j \lambda_j^4(\tilde{T}_{A,B}^g) &= \sum_{y,y'} B(y)B(y') |\mathcal{C}_3(A, g, \bar{g})(y, y')|^2 = \\ &= \sum_{x,x'} A(x)A(x') |\mathcal{C}_3(B, \bar{g}, g)(x, x')|^2. \end{aligned} \quad (29)$$

In the following lemma we find, in particular, all eigenvalues as well as all singularfunctions of operators  $T_{A,B}^{A-B}$ ,  $\tilde{T}_{A,B}^{A+B}$ .

**Lemma 14** *Let  $A, B \subseteq \mathbf{G}$  be finite sets,  $|B| \leq |A|$ ,  $D, S \subseteq \mathbf{G}$  be two sets such that  $A - B \subseteq D$ ,  $A + B \subseteq S$ . Then the main eigenvalues and singularfunctions of the operators  $T_{A,B}^D$ ,  $\tilde{T}_{A,B}^S$  equal  $\lambda_0 = (|A||B|)^{1/2}$ , and*

$$v_0(y) = B(y)/|B|^{1/2}, \quad \text{and} \quad u_0(x) = A(x)/|A|^{1/2},$$

correspondingly. All other singular values equal zero.

**Proof.** Using formulas (23), (24) it is easy to see that

$$(T_{A,B}^D(T_{A,B}^D)^*B)(y) = B(y) \sum_{y' \in B} \mathcal{C}_3(A, D, D)(-y, -y') = |A||B|B(y),$$

$$(\tilde{T}_{A,B}^S(\tilde{T}_{A,B}^S)^*B)(y) = B(y) \sum_{y' \in B} \mathcal{C}_3(A, S, S)(y, y') = |B|B(y)(A \circ S)(y) = |A||B|B(y).$$

Thus  $v_0(y) = B(y)/|B|^{1/2}$  and  $\lambda_0 = (|A||B|)^{1/2}$ . It follows that

$$u_0(x) = A(x)(|A|^{1/2}|B|)^{-1} \sum_y B(y)D(x-y) = A(x)/|A|^{1/2},$$

and

$$u_0(x) = A(x)(|A|^{1/2}|B|)^{-1}(B \circ S)(x) = A(x)/|A|^{1/2},$$

correspondingly for  $T_{A,B}^D$ ,  $\tilde{T}_{A,B}^S$ . By (27), we have

$$\sum_{j=0}^{|B|-1} \lambda_j^2 = |A||B|.$$

Thus all other singular values equal zero. □

Now we adapt the arguments from [25], see Proposition 28.

**Lemma 15** *Let  $A, B \subseteq \mathbf{G}$  be finite sets,  $D, S \subseteq \mathbf{G}$  be two sets such that  $A - B \subseteq D$ ,  $A + B \subseteq S$ . Suppose that  $\psi$  be a function on  $\mathbf{G}$ . Then*

$$|A|^2 \sigma^2(\psi, B) \leq \mathbf{E}_3(A, B) \sigma(\psi^2, D), \quad (30)$$

and

$$|A|^2 \sigma^2(\psi, B) \leq \mathbf{E}_3(A, B) \sigma(\psi^2, S). \quad (31)$$

**Proof.** Let us prove (31), the proof of (30) is similar. Denote by  $\lambda_j$  the singular values of  $T_{A,B}^S$  and by  $u_j, v_j$  the correspondent eigenfunctions. By Lemma 14, we, clearly, get

$$S(x+y)A(x)B(y) = \sum_{j=0}^{|B|-1} \lambda_j u_j(x) v_j(y) = \lambda_0 u_0(x) v_0(y).$$

Using Lemma 14 once more, we have

$$\sum_{x \in A} \sum_{y, z \in B} S(x+y) S(x+z) \psi(y-z) = \lambda_0^2 \sum_{y, z} \psi(y-z) v_0(y) v_0(z) = |A| \sigma(\psi, B).$$

But

$$\sum_{x \in A} \sum_{y, z \in B} S(x+y) S(x+z) \psi(y-z) = \sum_{\alpha, \beta} S(\alpha) S(\beta) \psi(\alpha - \beta) \mathcal{C}_3(-A, B, B)(\alpha, \beta).$$

By Cauchy–Schwarz, we obtain

$$|A|^2 \sigma^2(\psi, B) \leq \mathbf{E}_3(A, B) \sum_{\alpha, \beta} S(\alpha) S(\beta) \psi^2(\alpha - \beta) = \mathbf{E}_3(A, B) \sigma(\psi^2, S) \quad (32)$$

as required.  $\square$

**Corollary 16** *For any  $A, B \subseteq \mathbf{G}$  the following holds*

$$|A|^2 \mathbf{E}_{3/2}^2(B) \leq \mathbf{E}_3(A, B) \mathbf{E}(B, A \pm B) \leq \mathbf{E}_3^{1/3}(A) \mathbf{E}_3^{2/3}(B) \mathbf{E}(B, A \pm B). \quad (33)$$

This inequality was obtained in [15].

**Corollary 17** *For any  $A \subseteq \mathbf{G}$  the following holds*

$$|A|^6 \leq \mathbf{E}_3(A) \cdot \sum_{x \in A-A} ((A \pm A) \circ (A \pm A))(x).$$

That is an inequality from [21].

Now we need in more symmetric version of the operators above.

Let  $g : \mathbf{G} \rightarrow \mathbb{C}$  be a function, and  $A \subseteq \mathbf{G}$  be a finite set. By  $T_A^g$  denote the matrix

$$T_A^g(x, y) = g(x - y)A(x)A(y), \quad (34)$$

and by  $\tilde{T}_A^g(x, y)$  the matrix

$$\tilde{T}_A^g(x, y) = g(x + y)A(x)A(y). \quad (35)$$

General theory of such operators was developed in [24], and applications can be found in [22], [24], [25], [26]. Here we describe the simplest properties of matrices  $T_A^g$  and  $\tilde{T}_A^g$ . It is easy to see that  $T_A^g$  is hermitian iff  $\overline{g(-x)} = g(x)$  and  $\tilde{T}_A^g$  is hermitian iff  $g$  is a real function. Below we will deal with just hermitian operators with real functions  $g$ . In the case we arrange the eigenvalues in order of magnitude

$$|\mu_0(T_A^g)| \geq |\mu_1(T_A^g)| \geq \dots \geq |\mu_{|A|-1}(T_A^g)|,$$

and similar for  $\tilde{T}_A^g$ . We call  $\mu_0$  the main eigenvalue and the correspondent eigenfunction as the main eigenfunction. By Lemma 7 the following holds

$$\sum_j \mu_j(T_A^g) = g(0)|A| \quad \text{and} \quad \sum_j \mu_j(\tilde{T}_A^g) = \sum_x A(x)g(2x). \quad (36)$$

Further, in the case of hermitian (normal)  $T_A^g$ ,  $\tilde{T}_A^g$ , we get

$$\sum_j |\mu_j(T_A^g)|^2 = \sum_z |g(z)|^2 (A \circ A)(z) \quad \text{and} \quad \sum_j |\mu_j(\tilde{T}_A^g)|^2 = \sum_x |g(z)|^2 (A * A)(z). \quad (37)$$

Let also  $f_0, f_1, \dots, f_{|A|-1}$  be the sequence of correspondent eigenfunctions. Some results on the eigenfunctions can be found in [25].

Of course, the eigenvalues of operators  $T_{A,A}^g$  and the eigenvalues of operators  $T_A^g$  are connected by a simple formula  $\lambda_j(T_{A,A}^g) = |\mu_j(T_{A,A}^g)|$ , provided by  $T_A^g$  is a hermitian operator. The same formula holds for  $\tilde{T}_{A,A}^g$ ,  $\tilde{T}_A^g$ .

**Example 18** *One of the main ideas of using the operators of such sort is an attempt to find additively better subsets of  $A$  than the whole set  $A$ . A typical example here is the following. Let  $A = H \sqcup \Lambda \subseteq \mathbb{F}_p^n$ , where  $H$  is a subspace and  $\Lambda$  is a dissociated set (basis). Suppose that  $|H| \gg |A|^{2/3}$ ,  $|H| \ll |A|$ . Then  $E(A) \sim E(H)$  and  $A$  is not the main eigenfunction of the operator  $T_A^{A \circ A}$  because of  $E(A)/|A| < E(H)/|H| \leq E(A, H)/|H|$ . Thus, the main eigenfunction "sits" on  $H$  not on whole  $A$  in the case. Another idea of the operators method is an attempt to use "local" analysis on  $A$  in contrast to Fourier transformation method which is defined on the whole group  $\mathbf{G}$ .*

We have an analog of Lemma 14 with a similar proof.

**Lemma 19** *Let  $A \subseteq \mathbf{G}$  be a finite set,  $D, S \subseteq \mathbf{G}$  be two sets such that  $A - A \subseteq D$ ,  $A + A \subseteq S$ . Then  $T_A^D$ ,  $\tilde{T}_A^S$  have  $\mu_0 = |A|$ ,  $f_0(x) = A(x)/|A|^{1/2}$  and all other eigenvalues equal zero.*

**Proof.** It is easy to see that  $\mu_0 = |A|$  and  $f_0 = A(x)/|A|^{1/2}$  in both cases. Further by formulas (36), (37), we obtain

$$\sum_j \mu_j = |A| \quad \text{and} \quad \sum_j |\mu_j|^2 = |A|^2$$

for eigenvalues of  $T_A^D$  and  $\tilde{T}_A^S$ . Thus all other eigenvalues of both operators equal zero.  $\square$

**Remark 20** *If we take  $A = B$  in (30), (31), the function  $\psi$  equals  $\psi(x) = (A \circ A)(x)/(D \circ D)(x)$  or  $\psi(x) = (A \circ A)(x)/(S \circ S)(x)$  then we get*

$$\sum_{x \in D} \frac{(A \circ A)^2(x)}{(D \circ D)(x)} \leq \frac{E_3(A)}{|A|^2},$$

and

$$\sum_{x \in D} \frac{(A \circ A)^2(x)}{(S \circ S)(x)} \leq \frac{E_3(A)}{|A|^2},$$

*A little bit sharper inequality of such form was obtained in [25].*

Besides formulas (36), (37) there are some interesting relations between eigenvalues  $\mu_\alpha(T_A^g)$  and eigenfunctions  $f_\alpha$  of our operators. By  $g_\alpha$  denote the mean of the correspondent eigenfunction, that is  $g_\alpha = \sum_x f_\alpha(x)$ . We formulate our result for  $T_A^g$ . Of course, for  $\tilde{T}_A^g$  similar statement holds.

**Proposition 21** *Let  $g : \mathbf{G} \rightarrow \mathbb{C}$  be a function such that  $\overline{g(-x)} = g(x)$ . Then*

$$\sum_\alpha \mu_\alpha |g_\alpha|^2 = \sum_x g(x)(A \circ A)(x), \quad (38)$$

$$\sum_\alpha |\mu_\alpha|^2 |g_\alpha|^2 = \sum_{x \in A} |(g \circ A)(x)|^2. \quad (39)$$

*and if  $g$  is a real even nonnegative function then*

$$\sum_\alpha \mu_\alpha |\mu_\alpha g_\alpha|^2 \geq \frac{1}{|A|^2} \left( \sum_x g(x)(A \circ A)(x) \right)^3. \quad (40)$$

**Proof.** Formula (38) follows from the definition of the operator  $T_A^g$ , because of  $\langle T_A^g A, A \rangle = \sum_x g(x)(A \circ A)(x)$ . To prove (39) note that

$$\mu_\alpha f_\alpha(x) = A(x)(g * f_\alpha)(x).$$

Thus

$$\mu_\alpha g_\alpha = \sum_x A(x)(g * f_\alpha)(x). \quad (41)$$

Taking square of (41), summing over  $\alpha$  and using orthogonality of  $f_\alpha$ , we get

$$\begin{aligned} \sum_{\alpha} |\mu_{\alpha}|^2 |g_{\alpha}|^2 &= \sum_{\alpha} \sum_{x, x' \in A} f_{\alpha}(z) \overline{f_{\alpha}(z')} g(x-z) \overline{g(x'-z')} = \sum_{x, x' \in A} \sum_{z \in A} g(x-z) \overline{g(x'-z)} = \\ &= \sum_{x \in A} |(g \circ A)(x)|^2. \end{aligned}$$

To obtain (40) recall a useful inequality of A. Carbery [5] (see also [6]), namely,

$$\langle T f_1, f_2 \rangle^3 \leq \|f_1\|_3^3 \|f_2\|_3^3 \cdot \sum_{x, y} T(x, y) \left( \sum_a T(x, a) \right) \left( \sum_b T(b, y) \right) \quad (42)$$

which holds for  $T, f_1, f_2 \geq 0$ . Because of

$$T_A^g(x, y) = \sum_{\alpha} \mu_{\alpha} f_{\alpha}(x) \overline{f_{\alpha}(y)}$$

substitution  $T = T_A^g$  and  $f_1 = f_2 = A$  into (42) gives us

$$\begin{aligned} \left( \sum_x g(x) (A \circ A)(x) \right)^3 &\leq |A|^2 \cdot \sum_{x, y} \sum_{\alpha} \mu_{\alpha} f_{\alpha}(x) \overline{f_{\alpha}(y)} \left( \sum_{\beta} \overline{\mu_{\beta} f_{\beta}(x)} g_{\beta} \right) \left( \sum_{\gamma} \overline{\mu_{\gamma} g_{\gamma}} f_{\gamma}(y) \right) = \\ &= |A|^2 \cdot \sum_{\alpha} \mu_{\alpha} |\mu_{\alpha} g_{\alpha}|^2. \end{aligned}$$

This completes the proof.  $\square$

Let  $t$  be a positive integer. By  $(T_A^g)^{\otimes}$  denote the operator  $T_{A^{\otimes}}^{g^{\otimes}}$ , where tensor power of the functions  $g$  and  $A$  is taken  $t$  times. We will call the obtained operator as  $t$ -tensor power of  $T_A^g$ . Of course, if  $T_A^g$  is hermitian then  $(T_A^g)^{\otimes}$  is also hermitian. Let us prove a result on tensor powers of operators  $T_A^g$ .

**Lemma 22** *Let  $t$  be a positive integer and  $g : \mathbf{G} \rightarrow \mathbb{C}$  be a function such that  $\overline{g(-x)} = g(x)$ . Then eigenvalues and eigenfunctions of  $t$ -tensor power  $(T_A^g)^{\otimes}$  coincide with all  $t$ -products of eigenvalues and eigenfunctions of the operator  $T_A^g$ . In particular  $\mu_0((T_A^g)^{\otimes}) = \mu_0^t(T_A^g)$ .*

**Proof.** Because of  $\overline{g(-x)} = g(x)$  the operator  $T_A^g$  is hermitian. Let  $\{f_{\alpha}\}$ ,  $\alpha \in [|A|]$  be the family of all orthonormal eigenfunctions of  $T_A^g$ . Using formula (14) and the definition of the operator  $T_A^g$  it is easy to check that  $|A|^t$  products of the form  $\prod_{j=1}^{|A|} f_{\alpha_j}$ ,  $\alpha_j \in \{0, 1, \dots, |A| - 1\}$  are orthonormal eigenfunctions of  $(T_A^g)^{\otimes}$ . This completes the proof.  $\square$

In terms of the eigenvalues it is very natural to formulate structural results from papers [22], [25]. Here we give for example a variant of Theorem 56 from [25].

**Theorem 23** Let  $A \subseteq \mathbf{G}$  be a set,  $\mu_0 = T_A^{A \circ A}$ , and  $E_3(A) = M\mu_0^2$ . Suppose that  $M \leq \mu_0/(6|A|)$ . Then there is a real number  $r$

$$1 \leq r \leq \frac{1}{|A|} \max_{x \neq 0} (A \circ A)(x) \cdot \frac{|A|^2}{\mu_0} M^{1/2} \leq \frac{|A|^2}{\mu_0} M^{1/2}, \quad (43)$$

and a set  $A' \subseteq A$  such that

$$|A'| \gg M^{-23/2} r^{-2} \log^{-9} |A| \cdot |A|, \quad (44)$$

and

$$|nA' - mA'| \ll (M^9 \log^6 |A|)^{7(n+m)} r^{-1} M^{1/2} \frac{|A|^2}{\mu_0} |A'| \quad (45)$$

for every  $n, m \in \mathbb{N}$ .

Recall a lemma from [25].

**Lemma 24** Let  $A \subseteq \mathbf{G}$  be a set, and  $g$  be a nonnegative function,  $\mu_0 = \mu_0(T_A^g)$ . Then

$$|A| \geq \left( \sum_x f_0(x) \right)^2 \geq \max \left\{ \frac{\mu_0}{\|g\|_\infty}, \frac{\mu_0^2}{\|g\|_2^2} \right\}, \quad (46)$$

and

$$\|f_0\|_\infty \leq \frac{\|g\|_2}{\mu_0}. \quad (47)$$

If  $\widehat{g} \geq 0$  then

$$\|f_0\|_\infty \leq \frac{\|g_1\|_2}{\mu_0^{1/2}}, \quad (48)$$

where  $g = g_1 \circ \overline{g}_1$ .

Operator  $T_A^{A \circ A}$  is the simplest example of nonnegative defined operator on a set  $A$ . On the other hand it is connected with the additive energy, because of  $|A|^{-1} E(A) \leq \mu_0(T_A^{A \circ A})$  by Theorem 8, say. Thus it is natural to try to obtain some estimates on the main eigenvalue of the operator. We apply Lemma 24 to do this. Another lower bounds for  $\mu_0(T_A^{A \circ A})$  are contained in Theorem 43 of section 7.

**Corollary 25** For any  $A \subseteq \mathbf{G}$  the following holds

$$\mu_0(T_A^{A \circ A}) \geq \max_{g \geq 0} \frac{\mu^3(T_A^g)}{\|g\|_2^2 \cdot \|g\|_\infty}. \quad (49)$$

**Proof.** Let  $\mu = \mu_0(\mathbf{T}_A^g)$  and  $f = f_0$  be the correspondent eigenfunction. Instead of (49) we prove even stronger inequality, namely, the same lower bound for  $\langle \mathbf{T}_A^{A \circ A} f_0, f_0 \rangle$ . We have

$$\mu f(x) = A(x)(g * f)(x).$$

Thus

$$\mu^2 \left( \sum_x f(x) \right)^2 \leq \left( \sum_x g(x)(f \circ A)(x) \right)^2 \leq \|g\|_2^2 \mathbf{E}(A, f) \leq \|g\|_2^2 \mu_0(\mathbf{T}_A^{A \circ A}).$$

Applying estimate (46) of Lemma 24, we get

$$\left( \sum_x f(x) \right)^2 \geq \frac{\mu}{\|g\|_\infty}$$

and the result follows.  $\square$

We conclude the section recalling a result from [25], Proposition 22 (or see the proof of Proposition 21).

**Proposition 26** *Let  $A \subseteq \mathbf{G}$  be a set,  $g_1, g_2$  be even real functions and  $\{f_\alpha\}$  be the eigenfunctions of the operator  $\mathbf{T}_A^{g_1}$ . Then*

$$\sum_{x, y, z \in A} g_1(x - y)g_1(x - z)g_2(y - z) = \sum_{\alpha=0}^{|A|-1} \mu_\alpha^2(\mathbf{T}_A^{g_1}) \cdot \langle \mathbf{T}_A^{g_2} f_\alpha, f_\alpha \rangle.$$

## 5 Convex sets and sets with small multiplicative doubling

Now we apply technique from section 4 to obtain new upper bounds for the additive energy of some families of sets. Let us begin with the convex subsets of  $\mathbb{R}$ .

**Theorem 27** *Let  $A \subseteq \mathbb{R}$  be a convex set. Then*

$$\mathbf{E}(A) \ll |A|^{\frac{32}{13}} \log^{\frac{71}{65}} |A|. \quad (50)$$

**Proof.** Let  $\mathbf{E} = \mathbf{E}(A) = |A|^3/K$ ,  $\mathbf{E}_3 = \mathbf{E}_3(A)$ ,  $L = \log |A|$ . Applying formula (20) of Lemma 12 with  $k = 1$ , we obtain

$$2^{-2}\mathbf{E} \leq \sum_{s : 2^{-1}|A|K^{-1} < |A_s| \leq cK} |A_s|^2, \quad (51)$$

where  $c > 0$  is an absolute constant. Put

$$D_j = \{s \in A - A : 2^{j-2}|A|K^{-1} < |A_s| \leq 2^{j-1}|A|K^{-1}\},$$

where  $j \in [l]$ ,  $2^l \leq 2cK^2|A|^{-1} \ll K^2|A|^{-1}$ . Thus by (51) the following holds

$$2^{-2}\mathbf{E} \leq \sum_{j=1}^l \sum_{s \in D_j} |A_s|^2.$$

By pigeonhole principle, we find  $j \in [l]$  such that

$$2^{-2}l^{-1}\mathbf{E} \leq \sum_{s \in D_j} |A_s|^2 \leq |D_j|(2^{j-1}|A|K^{-1})^2. \quad (52)$$

Put  $D = D_j$ ,  $\Delta = 2^{j-1}|A|K^{-1}$ , and  $g(x) = (A \circ A)(x)D(x)$ . Consider the operators  $T_1 = T_A^g$ ,  $T_2 = T_{A,D}^A$  and  $T_3 = T_A^{A \circ A}$ . Clearly, all elements of matrices  $T_1, (T_2)^*T_2$  does not exceed elements of  $T_3$  and the operator  $T_3$  is nonnegative defined. By formula (52), we have

$$\frac{\mathbf{E}}{4l|A|} \leq \mu_0(T_1). \quad (53)$$

Similarly,

$$\frac{\mathbf{E}}{4l|A|} \leq \mu_0(T_1) \leq \langle T_3 f_0, f_0 \rangle, \quad (54)$$

where  $f_0 \geq 0$  is the main eigenfunction of the operator  $T_1$ . Applying Proposition 26 with  $A = A$ ,  $g_1 = g$ ,  $g_2 = A \circ A$ , we obtain

$$\mu_0^3(T_1) \leq \sum_{x,y,z \in A} g(x-y)g(x-z)(A \circ A)(y-z)$$

because of the operator  $T_3$  is nonnegative defined. Further

$$\mu_0^3(T_1) \leq \sum_{\alpha, \beta} g(\alpha)g(\beta)(A \circ A)(\alpha - \beta)\mathcal{C}_3(A)(\alpha, \beta). \quad (55)$$

The summation in (55) can be taken over  $\alpha, \beta$  such that

$$(A \circ A)(\alpha - \beta) \geq \frac{\mathbf{E}^2}{32L^2|A|^3\mathbf{E}_3^{1/2}} := d.$$

Indeed, otherwise by formulas (23) and (28), we get

$$\begin{aligned} \mu_0^3(T_1) &< d\Delta^2 \cdot \sum_{\alpha, \beta} D(\alpha)D(\beta)\mathcal{C}_3(A)(\alpha, \beta) = d\Delta^2 \cdot \langle T_2(T_2)^*D, D \rangle \leq \\ &\leq d\Delta^2|D|\mu_0(T_2(T_2)^*) \leq d\Delta^2|D|\mathbf{E}_3^{1/2} \end{aligned}$$

and we obtain a contradiction in view of the definition of the set  $D$  and inequality (53). Thus

$$2^{-1}\mu_0^3(T_1) \leq \sum_{\alpha, \beta : (A \circ A)(\alpha - \beta) \geq d} g(\alpha)g(\beta)(A \circ A)(\alpha - \beta)\mathcal{C}_3(A)(\alpha, \beta).$$



By Cauchy–Schwartz inequality and Lemma 12, we get

$$\begin{aligned} \mu_0^6(T_1) &\ll E_3 \sum_{\alpha \in D, \beta \in D : (A \circ A)(\alpha - \beta) \geq d} (A \circ A)^2(\alpha)(A \circ A)^2(\beta)(A \circ A)^2(\alpha - \beta) \\ &\ll |A|^3 L \Delta^3 \sum_{\alpha, \beta : (A \circ A)(\alpha - \beta) \geq d} D(\alpha)(A \circ A)(\alpha)D(\beta)(A \circ A)^2(\alpha - \beta) = |A|^3 L \Delta^3 \cdot \sigma. \end{aligned} \quad (56)$$

We estimate the quantity  $\sigma$  in two different ways. As above write

$$S_i = \{x : 2^{i-1}d < (A \circ A)(x) \leq 2^i d\}. \quad (57)$$

Clearly, by Lemma 12, we get  $|S_i| \ll |A|^3/(2^i d)^3$ . So for some  $i$

$$\sigma \ll L \sum_{\alpha, \beta, \alpha - \beta \in S_i} D(\alpha)(A \circ A)(\alpha)D(\beta)(A \circ A)^2(\alpha - \beta) = L\sigma_* \quad (58)$$

and your task is to estimate  $\sigma_*$ . We put  $\tau = 2^i d$  and write  $S_\tau$  for  $S_i$ . First of all by Lemma 12, we have

$$\sigma_* \ll \Delta \tau E(D, A) \ll \Delta \tau |A| |D|^{3/2}. \quad (59)$$

Second of all, applying the same lemma twice and also the estimate  $|S_\tau| \ll |A|^3/\tau^3$ , we obtain

$$\sigma_* \ll \tau^2 \sum_{\alpha} (A \circ A)(\alpha)(D \circ S_\tau)(\alpha) \ll \tau^2 |A| |D|^{3/4} |S_\tau|^{3/4} \ll \tau^{-1/4} |A|^{13/4} |D|^{3/4}. \quad (60)$$

Combining estimates (59), (60) and optimizing over  $\tau$ , we derive

$$\sigma_* \ll \Delta^{1/5} |A|^{14/5} |D|^{9/10}. \quad (61)$$

Returning to (56), recalling (53), (54), and substituting the last formula into (58), we get

$$\frac{E^6}{|A|^6 L^6} \ll \mu_0^6(T_1) \ll |A|^3 L^2 \Delta^3 \cdot \Delta^{1/5} |A|^{14/5} |D|^{9/10}.$$

Accurate computations, using (52) show

$$\left( \frac{E}{|A|L} \right)^{51/10} \ll |A|^{29/5+9/10} L^2 \Delta^{7/5}.$$

Applying estimate  $\Delta \ll K$  after some calculations we obtain the result. This completes the proof.  $\square$

**Corollary 28** *Let  $A \subseteq \mathbb{Z}$  be a convex set and*

$$P_A(\theta) = \sum_{a \in A} e^{2\pi i a \theta}.$$

*Then*

$$\int_0^{2\pi} |P_A(\theta)|^4 d\theta \ll |A|^{\frac{32}{13}} \log^{\frac{71}{65}} |A|.$$

**Remark 29** *The argument from the proof of Theorem 27 is quite tight modulo our current knowledge of convex sets. Indeed, if one put  $\tau = \Delta = K$  and, hence, by Lemma 12 we have  $|D| \ll |A|^3/K^3$  then the estimate  $K \gg |A|^{7/13-}$  is obtained exactly. The same situation takes place in the case of multiplicative subgroups of  $\mathbb{Z}/p\mathbb{Z}$ ,  $p$  is a prime number (see [25]), where the choice  $\tau = \Delta = K$  gives  $K \gg |A|^{5/9-}$ .*

Probably, using similar arguments one can obtain new upper bounds for  $T_k(A)$  as it was done in [25]. We do not make such calculations. For  $T_k(A)$  weighted Szemerédi–Trotter theorem would provide better bounds, probably.

Now we formulate a general result concerning the additive energy of sets with small multiplicative doubling.

**Theorem 30** *Let  $A \subseteq \mathbb{R}$  be a set. Suppose that  $|AA| = M|A|$ ,  $M \geq 1$ . Then*

$$E(A) \ll (M \log M)^{\frac{14}{13}} |A|^{\frac{32}{13}} \log^{\frac{71}{65}} |A|. \quad (62)$$

**Proof.** Let  $E = E(A) = |A|^3/K$ ,  $E_3 = E_3(A)$ ,  $L = \log |A|$ . By Lemma 13, we have  $E_3(A) \ll (M \log M)^2 \cdot |A|^3 \log |A|$ . Thus  $E_3(A)$  is small for small  $M$  and we can apply the arguments from the proof of Theorem 27. Using the first estimate of Lemma 13, we obtain

$$2^{-2}E \leq \sum_{j=1}^l \sum_{s \in D_j} |A_s|^2,$$

where

$$D_j = \{s \in A - A : 2^{j-2}|A|K^{-1} < |A_s| \leq 2^{j-1}|A|K^{-1}\},$$

$c > 0$  is an absolute constant and  $j \in [l]$ ,  $2^l \ll (M \log M)^2 K^2 |A|^{-1}$ . By pigeonhole principle, we find  $j \in [l]$  such that

$$2^{-2}l^{-1}E \leq \sum_{s \in D_j} |A_s|^2. \quad (63)$$

Put  $D = D_j$ ,  $\Delta = 2^{j-1}|A|K^{-1}$ , and  $g(x) = (A \circ A)(x)D(x)$ . After that apply arguments in lines (52)–(56), considering the operators  $T_1 = T_A^g$ ,  $T_2 = T_{A,D}^A$ ,  $T_3 = T_A^{A \circ A}$ , and also the upper bound for  $E_3$ , we have

$$\begin{aligned} \mu_0^6(T_1) &\ll E_3 \sum_{\alpha \in D, \beta \in D : (A \circ A)(\alpha - \beta) \geq d} (A \circ A)^2(\alpha)(A \circ A)^2(\beta)(A \circ A)^2(\alpha - \beta) \\ &\ll |A|^3 M^2 (\log M)^2 L \Delta^3 \sum_{\alpha, \beta : (A \circ A)(\alpha - \beta) \geq d} D(\alpha)(A \circ A)(\alpha)D(\beta)(A \circ A)^2(\alpha - \beta) = \\ &= |A|^3 M^2 (\log M)^2 L \Delta^3 \cdot \sigma. \end{aligned}$$

As is Theorem 27 the number  $d$  can be taken as  $d = \frac{E^2}{32L^2|A|E_3^{1/2}}$ . Using a consequence of the first estimate of Lemma 13, namely,  $|S_i| \ll (M \log M)^2 |A|^3 / (d^3 2^{3i})$ , the second bound from Lemma 13, and the arguments from lines (57)—(61), we obtain

$$\begin{aligned} \sigma &\ll \min_{\tau} \{ \Delta \tau |A| |D|^{3/2} (M \log M), \tau^{-1/4} |A|^{13/4} |D|^{3/4} (M \log M)^{5/2} \} \\ &\ll \Delta^{1/4} |A|^{14/5} |D|^{9/10} (M \log M)^{11/5}. \end{aligned}$$

Thus

$$\frac{E^6}{|A|^6 L^6} \ll |A|^3 L^2 \Delta^3 (M \log M)^2 \cdot \Delta^{1/5} |A|^{14/5} |D|^{9/10} (M \log M)^{11/5}.$$

Accurate computations as is Theorem 27 show

$$\left( \frac{E}{|A|L} \right)^{51/10} \ll |A|^{29/5+9/10} L^2 \Delta^{7/5} (M \log M)^{21/5}.$$

Using estimate  $\Delta \ll K(M \log M)^2$  after some calculations, we obtain

$$K \gg |A|^{7/13} L^{-71/65} (M \log M)^{-14/13}$$

as required.  $\square$

It is easy to check that Theorem 30 gives better bound for the additive energy than the bound from Lemma 13, namely  $E(A) \ll M \log M |A|^{5/2}$  if, roughly,  $M \ll |A|^{1/2-}$ .

In [25] the following theorem of the same type was obtained.

**Theorem 31** *Let  $A \subseteq \mathbb{R}$  be a set, and  $\varepsilon \in [0, 1)$  be a real number. Suppose that  $|AA| = M|A|$ ,  $M \geq 1$ , and*

$$|\{x \neq 0 : (A \circ A)(x) \geq |A|^{1-\varepsilon}\}| \ll (M \log M)^{\frac{5}{3}} |A|^{\frac{1}{6}-\frac{\varepsilon}{4}} \log^{\frac{5}{6}} |A|. \quad (64)$$

Then

$$E(A) \ll M \log M |A|^{\frac{5}{2}-\frac{\varepsilon}{12}} \log^{\frac{1}{2}} |A|. \quad (65)$$

Thus, our Theorem 30 is better than Theorem 31 if, roughly speaking,  $M \ll |A|^{\frac{1}{2}-\frac{13\varepsilon}{12}}$ . The advantage of Theorem 30 is the absence of  $\varepsilon$ , or, in other words, the absence more or less uniform upper bound for the convolution of  $A$ , of course.

Apply arguments of the proof of Theorem 30 for a new family of sets  $A$  with small quantity  $|A(A+1)|$ . Such sets were considered in [10], where the following lemma was proved.

**Lemma 32** *Let  $A, B \subseteq \mathbb{R}$  be two sets, and  $\tau \leq |A|, |B|$  be a parameter. Then*

$$|\{s \in AB : |A \cap sB^{-1}| \geq \tau\}| \ll \frac{|A(A+1)|^2 |B|^2}{|A| \tau^3}. \quad (66)$$

Lemma above implies that for any  $A \subseteq \mathbb{R}$  the following holds  $E^\times(A) \ll |A(A+1)||A|^{3/2}$ . Also in [10] a series of interesting inequalities were obtained. Here we formulate just one result.

**Theorem 33** *Let  $A \subseteq \mathbb{R}$  be a set. Then*

$$E^\times(A, A(A+1)), E^\times(A+1, A(A+1)) \ll |A(A+1)|^{5/2}.$$

In [25] all bounds of Theorem 33 were improved, provided by an analog of inequality (64) holds. We prove the following result, having no such condition.

**Corollary 34** *Let  $A \subseteq \mathbb{R}$  be a set,  $a \in \mathbb{R}$  be a number,  $|A(A+1)| = M|A|$ ,  $M \geq 1$ . Then*

$$E^\times(A, A+a) \ll M^{\frac{14}{13}} |A|^{\frac{32}{13}} \log^{\frac{71}{65}} |A| \quad (67)$$

*In particular*

$$E^\times(A) \ll M^{\frac{14}{13}} |A|^{\frac{32}{13}} \log^{\frac{71}{65}} |A| \quad (68)$$

**Proof.** Put  $A' = A+a$ , and now the convolution is the cardinality of the set  $\{a_1, a_2 \in A : x = a_1 a_2^{-1}\}$ . Lemma 32 implies that  $E_3^\times(A') \ll M^2 |A|^3 \log |A|$ . After that apply the arguments from the proof of Theorem 30.  $\square$

## 6 Structural results

Previous results of section 5 say, basically, that if  $E_3(A)$  is small and  $A$  has some additional properties, which show that  $A$  is "unstructured" in some sense then we can say something nontrivial about the additive energy of  $A$ . Now we formulate (see Theorem 35 below) a variant of the principle using just smallness of  $E_3(A)$  to show that  $A$  has a structured subset. There are several results of such type, see [22], [25]. Our new theorem is the strongest one in the sense that its has minimal requirements. From some point of view these type of statements can be called an optimal version of Balog–Szemerédi–Gowers theorem, see [22].

**Theorem 35** *Let  $A \subseteq \mathbf{G}$  be a set,  $E(A) = |A|^3/K$ , and  $E_3(A) = M|A|^4/K^2$ . Then there is a set  $A' \subseteq A$  such that*

$$|A'| \gg M^{-10} \log^{-15} M \cdot |A|, \quad (69)$$

*and*

$$|nA' - mA'| \ll (M^9 \log^{14} M)^{6(n+m)} K |A'| \quad (70)$$

*for every  $n, m \in \mathbb{N}$ . Moreover, if  $s \in (1, 3)$  is a real number, and we have the following condition  $E_s(A) = |A|^{s+1}/K^{s-1}$  then for all  $s \in (1, 3/2]$  there is a set  $A' \subseteq A$  such that*

$$|A'| \gg M^{-(14-4s)/(3-s)} (s-1)^{21} \log^{-21} (M(s-1)^{-1}) \cdot |A|, \quad (71)$$

and

$$|nA' - mA'| \ll (M^5(s-1)^{-20} \log^{20}(M(s-1)^{-1}))^{6(n+m)} K|A'|. \quad (72)$$

Finally, if  $s \in [3/2, 3)$  then there is a set  $A' \subseteq A$  such that

$$|A'| \gg M^{-(44-24s)/(3-s)} (3-s)^{21} \log^{-21} M \cdot |A|, \quad (73)$$

and

$$|nA' - mA'| \ll (M^{(45-25s)/(3-s)} (3-s)^{-20} \log^{20} M)^{6(n+m)} K|A'| \quad (74)$$

for every  $n, m \in \mathbb{N}$ .

**Proof.** Let  $E_s = E_s(A) = |A|^{s+1}/K^{s-1}$ ,  $E_3 = E_3(A)$ ,  $L = 2(3-s)^{-1} \log(4M(s-1)^{-1})$ . Because of  $E_3$  is small we can apply the arguments from the proof of Theorem 30. Write

$$D_j = \{x \in A - A : 2^{j-2}|A|K^{-1} < |A_x| \leq 2^{j-1}|A|K^{-1}\}.$$

Trivially

$$|D_j| (2^{j-2}|A|K^{-1})^3 \leq E_3,$$

and whence

$$|D_j| \ll E_3 / (|A|^3 K^{-3} 2^{3j}). \quad (75)$$

Thus

$$(s-1)E_s \ll \sum_{j=1}^l \sum_s |A_s|^s,$$

where  $l$  can be estimated as  $\log M^{1/(3-s)} = L$ . By pigeonhole principle we find  $j \in [l]$  such that

$$(s-1)L^{-1}E_s \ll \sum_{s \in D_j} |A_s|^s. \quad (76)$$

Put  $D = D_j$ ,  $\Delta = 2^{j-1}|A|K^{-1}$ , and  $g(x) = (A \circ A)^{s-1}(x)D(x)$ . From (76) it follows that

$$|D| \gg \frac{(s-1)|A|K}{LM^{s/(3-s)}} \quad (77)$$

and

$$\sum_{x \in D} (A \circ A)(x) \gg \frac{(s-1)|A|^2}{LM^{(s-1)/(3-s)}}. \quad (78)$$

After that apply arguments in lines (52)–(56), considering the operators  $T_1 = T_A^g$ ,  $T_2 = T_{A,D}^A$ ,  $T_3 = T_A^{A \circ A}$ . Using Corollary 25, we get

$$\langle T_3 f_0, f_0 \rangle \geq \frac{\mu_0^3(T_1)}{\|g\|_2^2 \|g\|_\infty} \gg \frac{|D|^2 \Delta^3}{|A|^3} := \sigma. \quad (79)$$

In the case  $s = 2$  as in Theorems 27, 30, we have  $\sigma \geq \mu_0(T_1)$ . Further, by Proposition 26, we obtain

$$(s-1)^6 \mu_0^4(T_1) \sigma^2 \ll E_3 \sum_{\alpha \in D, \beta \in D : (A \circ A)(\alpha - \beta) \geq d} (A \circ A)^{2s-2}(\alpha) (A \circ A)^{2s-2}(\beta) (A \circ A)^2(\alpha - \beta)$$

$$\ll E_3 \Delta^{4s-4} \sum_{x : (A \circ A)(x) \geq d} (D \circ D)(x) (A \circ A)^2(x), \quad (80)$$

where  $d$  can be taken as  $d = \frac{(s-1)^3 \sigma \mu_0(T_1)}{2^{13} L \Delta^{s-2} E_3^{1/2}}$  (and  $d = \frac{\mu_0^2(T_1)}{32|A|E_3^{1/2}}$  in the case  $s = 2$ ). Applying Cauchy–Schwartz inequality, we have

$$\sum_x (A \circ A)^2(x) (D \circ D)(x) \leq E_3^{2/3} \left( \sum_x (D \circ D)^3(x) \right)^{1/3} \leq E_3^{2/3} |D|^{1/3} E^{1/3}(D).$$

Put  $E(D) = \mu|D|^3$ . Recalling (80), we get

$$(s-1)^6 \mu_0^4(T_1) \sigma^2 \ll \left( \frac{M|A|^4}{K^2} \right)^{5/3} \Delta^{4s-4} |D|^{4/3} \mu^{1/3}. \quad (81)$$

We have  $\Delta \ll M^{1/(3-s)}|A|/K$ . First of all consider the case  $s = 2$ . In the situation the following holds  $\sigma \geq \mu_0(T_1)$ . Thus, an accurate calculations give

$$E(D) = \mu|D|^3 \gg \frac{|D|^3}{M^9 L^{14}}.$$

By Balog–Szemerédi–Gowers Theorem 11 there is  $D' \subseteq D$  such that  $|D'| \gg \mu|D|$  and  $|D' + D'| \ll \mu^{-6}|D'|$ . Plünnecke–Ruzsa inequality (see e.g. [28]) yields

$$|nD' - mD'| \ll \mu^{-6(n+m)} |D'|, \quad (82)$$

for every  $n, m \in \mathbb{N}$ . Using the definition of the set  $D = D_j$  and inequality (78) (recall that we are considering the case  $s = 2$ ), we find  $x \in \mathbf{G}$  such that

$$|(A - x) \cap D'| \gg \mu|A|L^{-1}M^{-1} \gg M^{-10}L^{-15} \cdot |A|. \quad (83)$$

Put  $A' = A \cap (D' + x)$ . Using (82), (83) and the definition of  $\Delta$ , we obtain for all  $n, m \in \mathbb{N}$

$$|nA' - mA'| \leq |nD' - mD'| \ll \mu^{-7(n+m)} |A||A'|\Delta^{-1} \ll \mu^{-6(n+m)} K|A'| \quad (84)$$

and the theorem is proved in the case  $s = 2$ .

Now take any  $s \in (1, 3)$ . Returning to (81), using (79) and making similar computations, we obtain

$$\left( \frac{(s-1)E_s}{L|A|} \right)^{20/3} \Delta^{10-20s/3} \ll |A|^{10/3} \mu^{1/3} \left( \frac{M|A|^4}{K^2} \right)^{5/3}. \quad (85)$$

Suppose that  $s \in (1, 3/2]$ . In the case

$$\mu \gg \frac{(s-1)^{20}}{M^5 L^{20}}$$

because of  $\Delta \gg |A|/K$ . After that repeat the arguments above. If  $s \in [3/2, 3)$  then (85) gives us

$$\left( \frac{(s-1)E_s}{L|A|} \right)^{20/3} (M^{1/(3-s)}|A|K^{-1})^{10-20s/3} \ll |A|^{10/3} \mu^{1/3} \left( \frac{M|A|^4}{K^2} \right)^{5/3}$$

because of  $\Delta \ll M^{1/(3-s)}|A|/K$ . Computations show

$$\mu \gg \frac{1}{M^{(45-25s)/(3-s)}L^{20}}.$$

After that repeat the arguments above once more. This completes the proof.  $\square$

Certainly, inequality (69) and the assumption  $E(A) = |A|^3/K$  of the Theorem 35 imply that  $|A' - A'| \gg_M K|A'|$ . Thus, we need in the multiple  $K$  in (74).

Of course, using the definition of the number  $\Delta$  more accurate one can improve estimates (69), (70) a little bit.

**Remark 36** *For every convex set Theorem 35 above easily gives a "nontrivial" estimate  $E(A) \ll |A|^{5/2-\varepsilon_0}$ , where  $\varepsilon_0 > 0$  is an absolute constant. Indeed, putting  $M = \log |A|$ , using formula (19) of Lemma 12 and the upper bound for the energy  $E_3(A)$  follows from the lemma, we have for the set  $A'$  from Theorem 35 that*

$$|A|^{7/4} \ll_M |A' + A' - A'| \ll_M |A|^4 E^{-1}(A)$$

*and the result follows. Applying more refine method from [21] one can get even simpler proof. Indeed, for so large  $A' \subseteq A$  we have  $|A|^{3/2+\varepsilon_1} \ll_M |A' - A'| \ll_M |A|^4 E^{-1}(A)$ ,  $\varepsilon_1 > 0$  is an absolute constant and again we obtain a lower bound for  $\varepsilon_0$ . Interestingly, that lower bounds for doubling constants give us upper bounds for the additive energy in the case.*

*The same proof takes place for multiplicative subgroups  $\Gamma \subseteq \mathbb{Z}/p\mathbb{Z}$ , where  $p$  is a prime number if one use Stepanov's method (see e.g. [13] or [27]) or combine Stepanov's method with recent lower bounds for the doubling constant from [20, 27, 25]. Note that an estimate of the sort  $E(\Gamma) \ll |\Gamma|^{5/2-\varepsilon}$  was known before and was obtained by another variant of the eigenvalues method, see [25]. On the other hand any multiplicative subgroup  $\Gamma$  of size  $|\Gamma| > p^\varepsilon$  is an additive basis of  $\mathbb{Z}/p\mathbb{Z}$  of order  $C(\varepsilon)$  (see e.g. [7], [3] and general sum-product inequalities in [17]). The fact that the same is true for sets with small multiplicative doubling was obtained in [4] (more precisely, Bourgain proved that Fourier coefficients of such sets are small in average) and this also implies that a "non-trivial" upper bound for  $E(\Gamma)$  holds.*

The arguments above allow replace the condition on  $E_3$  in Theorem 35 onto the same condition on  $E_4$  easily. By evenness the proof is simpler in the situation. General result of the same type with another constants was obtained in [22], see Theorem 54. We include the proof here because of the next important Theorem 38, which can be obtained by almost the same arguments.

**Theorem 37** *Let  $s \in [8/5, 4)$  be a real number,  $A \subseteq \mathbf{G}$  be a set,  $E_s(A) = |A|^{s+1}/K^{s-1}$ , and  $E_4(A) = M|A|^5/K^3$ . Then there is a set  $A' \subseteq A$  such that*

$$|A'| \gg M^{-(5s-5)/(4-s)}(4-s)^6 \log^{-6} M \cdot |A|, \quad (86)$$

*and*

$$|nA' - mA'| \ll (M^{(4s-4)/(4-s)}(4-s)^{-5} \log^5 M)^{6(n+m)} K|A'| \quad (87)$$

for every  $n, m \in \mathbb{N}$ . If  $s \in (1, 8/5]$  then there is a set  $A' \subseteq A$  such that

$$|A'| \gg M^{-3/(4-s)}(s-1)^6 \log^{-6}(M(s-1)^{-1}) \cdot |A|, \quad (88)$$

and

$$|nA' - mA'| \ll (M(s-1)^{-5} \log^5(M(s-1)^{-1}))^{6(n+m)} K |A'| \quad (89)$$

for every  $n, m \in \mathbb{N}$ .

*Proof.* Let  $E_4 = E_4(A)$ ,  $L = 2(4-s)^{-1} \log(4M(s-1)^{-1})$ . In terms of Theorem 35, we have

$$\begin{aligned} (s-1)^8 \mu_0^8(T_1) &\ll \left( \sum_{x,y,z,w \in A} g(x-y)g(y-z)g(z-w)g(w-x) \right)^2 = \\ &= \left( \sum_{\alpha,\beta,\gamma} C_4(A)(\alpha, \beta, \gamma) g^2(\alpha)g^2(\beta-\alpha)g^2(\gamma-\beta)g^2(\gamma) \right)^2 \ll \\ &\ll E_4 \cdot \sum_{\alpha,\beta,\gamma} g^2(\alpha)g^2(\beta-\alpha)g^2(\gamma-\beta)g^2(\gamma) \ll E_4 \cdot \Delta^{8s-8} E(D) \ll \frac{M|A|^5}{K^3} \cdot \Delta^{8s-8} E(D). \end{aligned}$$

Here  $g(x) = D(x)(A \circ A)^{s-1}(x)$ ,  $T_1 = T_A^g$ ,  $D = D_j$ ,  $\Delta = 2^{j-1}|A|K^{-1}$ . We have  $2^j \ll M^{1/(4-s)}$  and hence  $\Delta \ll M^{1/(4-s)}|A|K^{-1}$ . Note that the number  $j$  can be estimated by  $\log M^{1/(4-s)} = L$ . Put  $E(D) = \mu|D|^3$ . After some accurate calculations, we get for  $s \in [8/5, 4)$  that

$$\left( \frac{|A|^s}{K^{s-1}L} \right)^5 \ll \left( (s-1) \frac{|A|^s}{K^{s-1}L} \right)^5 \ll \frac{M|A|^5}{K^3} \Delta^{5s-8} |A|^3 \mu \ll \frac{M|A|^5}{K^3} (M^{1/(4-s)}|A|K^{-1})^{5s-8} |A|^3 \mu.$$

Whence

$$\mu \gg \frac{1}{M^{(4s-4)/(4-s)} L^5}.$$

After that repeat the arguments from lines (82)—(84) of the proof of Theorem 35.

If  $s \in (1, 8/5)$  then it is easy to see that

$$\mu \gg \frac{(s-1)^5}{ML^5}.$$

Repeating the arguments above gives the result. This concludes the proof.  $\square$

Of course Theorems 35, 37 can be generalized onto higher moments but such generalizations became weaker if one consider higher  $E_k$  because we should have deal with  $T_k(D)$  not  $E(D)$ .

Instead of this we take another characteristic of a set  $A$ , namely, its energy  $T_4(A)$  and obtain structural theorem in the situation. Similar result was obtained in [22], see Theorem 60. Theorem 2 from the introduction has the same form but weaker assumption.



**Theorem 38** *Let  $A \subseteq \mathbf{G}$  be a set,  $\mathbf{E}_{3/2}(A) = |A|^{5/2}/K^{1/2}$ , and  $\mathbf{T}_4(A) = M|A|^7/K^3$ . Then there is a set  $A' \subseteq A$  such that*

$$|A'| \gg \frac{|A|}{MK}, \quad (90)$$

and

$$\mathbf{E}(A') \gg \frac{|A'|^3}{M}. \quad (91)$$

*If  $s$  is a real number,  $s \in (1, 3/2]$  and we have the following condition  $\mathbf{E}_s(A) = |A|^{s+1}/K^{s-1}$  then there is a set  $A' \subseteq A$  such that*

$$|A'| \gg \frac{(s-1)^8 |A|}{MK \log^8 K}, \quad (92)$$

and

$$\mathbf{E}(A') \gg \frac{(s-1)^8 |A'|^3}{M \log^8 K}. \quad (93)$$

**Proof.** Let  $\mathbf{E}_4 = \mathbf{E}_4(A)$ ,  $\mathbf{T}_4 = \mathbf{T}_4(A)$ . In terms of Theorems 35, 37, we have

$$\begin{aligned} \mu_0^8(\mathbf{T}_1) &\ll \left( \sum_{x,y,z,w \in A} g(x-y)g(y-z)g(z-w)g(w-x) \right)^2 \ll \\ &\ll \mathbf{E}_4 \cdot \sum_{\alpha,\beta,\gamma} g^2(\alpha)g^2(\beta-\alpha)g^2(\gamma-\beta)g^2(\gamma) \ll \mathbf{E}_4 \cdot \mathbf{T}_4 \ll \mathbf{E}_4 \cdot \frac{M|A|^7}{K^3}. \end{aligned}$$

Here  $g(x) = (A \circ A)^{s-1}(x)$ ,  $\mathbf{T}_1 = \mathbf{T}_A^g$ . Put  $\mathbf{E}_4 = \mu|A|^5$ . First of all consider the case  $s = 3/2$ . After some calculations, we get

$$\mu \gg \frac{1}{MK}.$$

Clearly,

$$\sum_{s : |A_s| < 2^{-2}\mu|A|} \sum_t \mathbf{E}(A_s, A_t) \leq \sum_{s : |A_s| < 2^{-2}\mu|A|} \sum_t |A_s|^2 |A_t| < 2^{-2} \mathbf{E}_4.$$

Thus by Lemma 10 one has

$$\sum_{s,t : |A_s|, |A_t| \geq 2^{-2}\mu|A|} \mathbf{E}(A_s, A_t) \geq 2^{-1} \mathbf{E}_4. \quad (94)$$

Put

$$\nu := \max_{|A_s|, |A_t| \geq 2^{-2}\mu|A|} \frac{\mathbf{E}(A_s, A_t)}{|A_s|^{3/2} |A_t|^{3/2}}.$$

By (94), we have

$$2^{-1} \mathbf{E}_4 \leq \nu \sum_{s,t} |A_s|^{3/2} |A_t|^{3/2} = \nu \mathbf{E}_{3/2}^2(A)$$

and hence  $\nu \geq 2^{-1}\mu K$ . It follows that there are  $s, t$  such that  $|A_s|, |A_t| \geq 2^{-2}\mu|A|$  and

$$\mathbf{E}(A_s, A_t) \geq \nu|A_s|^{3/2}|A_t|^{3/2} \gg \frac{|A_s|^{3/2}|A_t|^{3/2}}{M}.$$

Applying Cauchy–Schwartz inequality, we obtain the result.

Now let  $s \in (1, 3/2]$ . In the case we put  $g(x) = D(x)(A \circ A)^{s-1}(x)$ , where the set  $D$  is defined as in Theorems 35, 37. Then the following holds

$$(s-1)^8 \left( \frac{\mathbf{E}_s(A)}{|A|L} \right)^8 \ll \mathbf{E}_4 \mathbf{T}_4 \Delta^{8s-12},$$

where  $\Delta \gg |A|/K$  and  $L \ll \log K$ . Hence

$$\mu \gg \frac{(s-1)^8}{ML^8 K}.$$

After that repeat the arguments above. This completes the proof.  $\square$

**Remark 39** All bounds of Theorem 38 are tight as an example  $A = H \dot{+} \Lambda \subseteq \mathbb{F}_2^n$  shows. Here  $H \leq \mathbb{F}_2^n$  as a subspace and  $\Lambda$  is a dissociated set (basis) (see also Example 42 from section 7). This set  $A$  corresponds to the case  $\alpha = 0$  in Theorem 2 from introduction. There are another more difficult examples which demonstrate the same.

**Remark 40** The proof of Theorem 38 gives, in particular, that

$$\left( \frac{\mathbf{E}_{3/2}(A)}{|A|} \right)^{2k} \leq \mathbf{E}_k(A) \mathbf{T}_k(A)$$

and

$$\left( \frac{\sum_{x \in D} (A \circ A)(x)}{|A|} \right)^{2k} \leq \mathbf{E}_k(A) \mathbf{T}_{k/2}(D) \quad (95)$$

for any sets  $A, D \subseteq \mathbf{G}$  and even positive  $k$ . Some particular case of the last formula appeared in [22], see Lemma 3 and also Remark 61.

**Remark 41** In Theorem 38 we have found a set  $A'$  with huge additive energy. Thus, by Balog–Szemerédi–Gowers Theorem there is a huge subset of  $A'$  with small doubling, similarly to Theorem 2 from introduction. Remark 39 shows that there is an example demonstrating sharpness of our theorem and having parameter  $\alpha = 0$ . It is easy to construct similar counterexample corresponding to the opposite case  $\alpha = (1 - \tau_0)/2$  in Theorem 2. Indeed, let  $H_1, \dots, H_k$ , where  $k = \lfloor K^{1/2} \rfloor$  be some totally disjoint subspaces of  $\mathbb{F}_2^n$  in the sense that  $|H_1 + \dots + H_k| = |H_1| \dots |H_k|$  (see e.g. [18]). Put  $A = \bigsqcup_{j=1}^k H_j$ . Then  $\mathbf{T}_t(A) \sim |A|^{2t-1}/K^{t-1}$ ,  $\mathbf{E}_s(A) \sim |A|^{s+1}/K^{s/2}$  but there is no any nontrivial  $X_j$  here. So, once more, in terms of  $\mathbf{E}_4(A)$  and  $\mathbf{T}_4(A)$  our Theorem 38 is the best possible (even for smaller  $\mathbf{E}_{3/2}(A)$ ) as the example above shows. Of course if we know something on “height” (see [1, 2] or section 7) of the set  $A$  then such  $X_j$  can appear (inequality (95) cast light to this slightly). We discuss the example with subspaces  $H_1, \dots, H_k$  in section 7.

The particular case when the parameter  $s$  equals 1 in the theorems of the section was considered in [25], see also [22].

## 7 Dual popular sets

Let  $k \geq 2$  be an integer and  $c \in (0, 1]$  be a real number. Given a set  $A \subseteq \mathbf{G}$ , we call a set  $\mathcal{P} \subseteq \mathbf{G}^{k-1}$  a  $(k, c)$ -dual (or just dual) to a set  $P \subseteq \mathbf{G}$  if

$$\begin{aligned} c\mathbf{E}_k^P(A) &\leq \sum_{x,y} P(x-y)A(x)A(y) \times \\ &\times \sum_{z_1, \dots, z_{k-1}} \mathcal{P}(z_1, \dots, z_{k-1})A(x+z_1) \dots A(x+z_{k-1})A(y+z_1) \dots A(y+z_{k-1}). \end{aligned} \quad (96)$$

In the same way we can define a  $(k, c)$ -dual set  $P \subseteq \mathbf{G}$  to a set  $\mathcal{P} \subseteq \mathbf{G}^{k-1}$  if

$$\begin{aligned} c\mathbf{E}_k^{\mathcal{P}}(A) &\leq \sum_{x,y} P(x-y)A(x)A(y) \times \\ &\times \sum_{z_1, \dots, z_{k-1}} \mathcal{P}(z_1, \dots, z_{k-1})A(x+z_1) \dots A(x+z_{k-1})A(y+z_1) \dots A(y+z_{k-1}). \end{aligned} \quad (97)$$

Of course a dual set is not unique and we write the fact that  $\mathcal{P}$  belongs to the family of dual sets of  $P$  as  $\mathcal{P} = P^*$  and vice versa.

It is easy to find a pair of dual sets, e.g. take for any  $P$  the set  $\mathcal{P} = A^{k-1} - \Delta_{k-1}(A)$  and for any  $\mathcal{P}$  the set  $P = A - A$ . Let us consider another examples. Let  $P \subseteq \mathbf{G}$  be a popular difference set in the sense that

$$P = \{z : |A_z|^{k-1} \geq \mathbf{E}_k(A)(2|A|^2)^{-1}\}.$$

Then

$$2^{-1}\mathbf{E}_k(A) \leq \mathbf{E}_k^P(A) = \sum_{z \in P} |A_z|^k = \sum_{z_1, \dots, z_k} A(z_1) \dots A(z_k) \mathcal{C}_{k+1}(P, A)(z_1, \dots, z_k). \quad (98)$$

If we put

$$\mathcal{P} = \{(z_1, \dots, z_{k-1}) : \mathcal{C}_k(A)(z_1, \dots, z_{k-1}) \geq \mathbf{E}_k(A)(4|A|^k)^{-1}\}$$

then

$$\begin{aligned} 2^{-2}\mathbf{E}_k(A) &\leq 2^{-1}\mathbf{E}_k^P(A) \leq \sum_{z_1, \dots, z_k} A(z_1) \dots A(z_k) \mathcal{P}(z_1 - z_k, \dots, z_{k-1} - z_k) \mathcal{C}_{k+1}(P, A)(z_1, \dots, z_k) = \\ &= \sum_{x,y} P(x-y)A(x)A(y) \sum_{z_1, \dots, z_{k-1}} \mathcal{P}(z_1, \dots, z_{k-1})A(x+z_1) \dots A(x+z_{k-1})A(y+z_1) \dots A(y+z_{k-1}). \end{aligned}$$

Thus, we have constructed  $(k, 1/2)$ -dual sets. In the same way we can start from the inequality

$$2^{-1}\mathbf{E}_k(A) \leq \mathbf{E}_k^{\mathcal{P}}(A) = \sum_{(z_1, \dots, z_{k-1}) \in \mathcal{P}} \mathcal{C}_k^2(A)(z_1, \dots, z_{k-1})$$

and after that define  $P$ , showing that  $P = \mathcal{P}^*$ . In these two examples  $P$  and  $\mathcal{P}$  are popular difference sets. Thus, we call the pair  $P, \mathcal{P}$  of the form as  $(k, 1/4)$ -popular dual sets. If we have

$$c\mathbf{E}_k(A) \leq \sum_{x,y} P(x-y)A(x)A(y) \times$$

$$\times \sum_{z_1, \dots, z_{k-1}} \mathcal{P}(z_1, \dots, z_{k-1}) A(x + z_1) \dots A(x + z_{k-1}) A(y + z_1) \dots A(y + z_{k-1}) \quad (99)$$

or, in other words, if  $P, \mathcal{P}$  are  $(k, c)$ -popular dual sets then, clearly,

$$cE_k(A) \leq \sum_{z \in P} |A_z|^k \quad \text{and} \quad cE_k(A) \leq \sum_{(z_1, \dots, z_{k-1}) \in \mathcal{P}} \mathcal{C}_k^2(A)(z_1, \dots, z_{k-1}).$$

So, there is a converse implication, in some sense.

Note also that if  $P, \mathcal{P}$  are  $(k, c)$ -popular dual sets then by formulas (14), (15) for any integer  $t$  tensor powers  $P^\otimes, \mathcal{P}^\otimes$  form  $(k, c^t)$ -popular dual sets for  $A^\otimes$ . Another examples of popular dual sets are level popular difference sets, that is

$$P_i = \{s : 2^{i-1}E_k(A)(2|A|^2)^{-1} < |A_s|^{k-1} \leq 2^iE_k(A)(2|A|^2)^{-1}\},$$

and

$$\mathcal{P}_j = \{(z_1, \dots, z_{k-1}) : 2^{j-1}E_k(A)(4|A|^k)^{-1} < \mathcal{C}_k(A)(z_1, \dots, z_{k-1}) \leq 2^jE_k(A)(4|A|^k)^{-1}\}$$

in the sense that there are  $i, j \in [L]$  such that  $P_i$  and  $\mathcal{P}_j$  are  $(k, 2^{-2}L^{-2})$ -popular dual sets, where  $L = L(A) = \log(4|A|^{k+1}E_k^{-1}(A))$ . In the case we define

$$\Delta = \Delta(P) = \Delta(A, P) = 2^iE_k(A)(2|A|^2)^{-1},$$

and

$$\Delta^* = \Delta(P^*) = \Delta(A, P^*) = 2^jE_k(A)(4|A|^k)^{-1}.$$

The situation  $k = 2$  is the most interesting. In the case there is a dual formula

$$\sum_{x, y} A(x)A(y)g(x - y)\mathcal{C}_3(h, A, A)(x, y) = \sum_{x, y} A(x)A(y)h(x - y)\mathcal{C}_3(g, A, A)(x, y), \quad (100)$$

where  $g, h : \mathbf{G} \rightarrow \mathbb{C}$  are any functions. The formula above shows that one has  $(P^*)^* = P$  in the sense that the set  $P$  is a popular dual from the family of all popular dual sets of  $P^*$ . We will write  $P^*$  instead of  $\mathcal{P}$  in the case and  $c$ -popular dual instead of  $(2, c)$ -popular dual.

**Example 42** Consider the set  $A$  from Remark 39. The set of popular differences of  $A$  naturally splits onto two sets

$$P_1 = H = \{x : |A_x| = |A|\} \quad \text{and} \quad P_2 = \{x \in (A - A) \setminus H : |A_x| = |H|\}.$$

It is easy to check that  $P_2 = P_1^*$  and vice versa. Note also that for  $s \geq 1$  one has

$$E_s(A) \sim |H||A|^s + |A|^2|H|^{s-1} \sim \begin{cases} |H||A|^s & \text{if } s \geq 2 \\ |A|^2|H|^{s-1} & \text{if } s < 2 \end{cases}$$

So, the sum over  $P_1$  in  $E_s(A)$  dominates for large  $s$  and the sum over  $P_2$  dominates for small  $s$ .

Now we prove the main result concerning properties of dual sets. The most interesting part is inequality (106), which gives a non-trivial relation between  $E(A)$  and  $E_s(A)$ ,  $s \in [1, 2]$ . Bounds (102)–(105) and (107) say that there is a connection between some characteristics of dual sets. As a consequence (see [1, 2] or corollary and proposition below), one can derive that for any "regular" (see exact formulation below) set  $A$  one can find a set  $Q \subseteq A - A$  such that  $E_Q(A) \gg E^{1-}(A) := |A|^{3-}/K$  and  $\sigma_Q(A) \gg |A|^{2-}/K^{1/2}$ .

Consider the hermitian operator

$$T(x, y) = A(x)A(y) \sum_{z_1, \dots, z_{k-1}} \mathcal{P}(z_1, \dots, z_{k-1}) A(x + z_1) \dots A(x + z_{k-1}) A(y + z_1) \dots A(y + z_{k-1}). \quad (101)$$

It is easy to see that  $T$  is nonnegative defined. By formula (25) for  $k = 2$  the operator coincide with  $(T_{A, \mathcal{P}^c}^A)^* T_{A, \mathcal{P}^c}^A$ .

**Theorem 43** *Let  $A \subseteq \mathbf{G}$  be a set. Suppose that the notation above takes place. Then there are two  $(k, 2^{-2}L^{-2})$ -popular dual sets  $P, \mathcal{P}$  such that*

$$\Delta \Delta^* \leq 16L\mu_0(T_A^{(A \circ A)^{k-1}}), \quad (102)$$

and

$$E_k^2(A) \leq 16L^2\mu_0(T_A^{(A \circ A)^{k-1}})\sigma_P(A)\sigma_{\mathcal{P}}(A), \quad (103)$$

$$E_k^2(A) \leq 2^8 L^3 \mu_0^2(T_A^{(A \circ A)^{k-1}})|P||\mathcal{P}|, \quad (104)$$

$$E_k^2(A) \leq 16L^4\sigma_P(A)\sigma_{\mathcal{P}}(A)\mu_0(T). \quad (105)$$

In the case  $k = 2$  for any  $s \in [1, 2]$  the following holds

$$E(A) \leq \mu_0(T_A^{A \circ A})^{1-s/2} E_s(A) \quad (106)$$

and if  $P^*$  is  $c$ -dual to  $P$  then

$$c^2 E_P^2(A) \leq \sigma_{P^*}(A) \cdot \sum_{x, y \in P} \mathcal{C}_3^2(A)(x, y). \quad (107)$$

**Proof.** Firstly, we prove an approximate formulas up to some logarithms and constants and after that we will use tensor trick (see e.g. [28]), replacing  $A$  by  $A^t$ , where  $t$  is an integer. By formula (14), we have  $E_s(A^t) = E_s^t(A)$  for any  $s$ . Applying Lemma 22, we get  $\mu_0(T_{A^{\otimes}}^{(A \circ A)^{\otimes}}) = \mu_0^t(T_A^{A \circ A})$ . Finally,  $L(A^t) \leq tL(A)$ , where  $L = \log(4|A|^{k+1}E_k^{-1}(A))$ .

We begin with (102). Let  $P = P_i$ ,  $\mathcal{P} = \mathcal{P}_j$  are  $(k, 2^{-2}L^{-2})$ -popular dual sets, with  $L$  defined above. By the definition of the operator  $T$  and formula (99), we have

$$E_k(A)2^{-2}L^{-2} \leq \sum_{\alpha} \mu_{\alpha}(T_A^P) \langle T f_{\alpha}, f_{\alpha} \rangle \leq \quad (108)$$

$$\leq \mu_0(T_A^P) \sum_{\alpha} \langle T f_{\alpha}, f_{\alpha} \rangle = \mu_0(T_A^P) \cdot \sum_{(z_1, \dots, z_{k-1}) \in \mathcal{P}} \mathcal{C}_k(A)(z_1, \dots, z_{k-1}), \quad (109)$$

where  $\{f_{\alpha}\}_{\alpha \in [|A|]}$  are the eigenfunctions of the operator  $T_A^P$ . Here we have used the fact that  $P$  is symmetric. Multiplying the last inequality by  $\Delta \Delta^*$  and using the definition of the sets  $P, \mathcal{P}$ , we get

$$\Delta \Delta^* \leq 16L\mu_0(T_A^{P(A \circ A)^{k-1}}) \leq 16L\mu_0(T_A^{(A \circ A)^{k-1}}).$$

Let us prove (103), (104) and (105). Multiplying (109) by  $\Delta \sigma_P(A)$ , we get (103). Combining (102) and (103), we have (104). Returning to (108) and using Cauchy–Schwartz inequality, we obtain

$$\mathbb{E}_k^2(A) 2^{-4} L^{-4} \leq \sum_{\alpha} \mu_{\alpha}^2(T_A^P) \cdot \sum_{\alpha} \langle T f_{\alpha}, f_{\alpha} \rangle^2.$$

Now applying formula (37) and Lemma 9, we have

$$\mathbb{E}_k^2(A) 2^{-4} L^{-4} \leq \sigma_P(A) \sum_{\alpha} \mu_{\alpha}^2(T) \leq \sigma_P(A) \mu_0(T) \sum_{\alpha} \mu_{\alpha}(T) = \sigma_P(A) \sigma_P(A) \mu_0(T)$$

and (105) is proved.

In the case  $k = 2$ , we have  $\tilde{\Delta} := \min\{\Delta, \Delta^*\} \leq (16L\mu_0(T_A^{A \circ A}))^{1/2}$ . Suppose that the minimum is attained at  $P^*$ , the opposite case can be considered similarly. Then

$$\mathbb{E}(A) \leq 4L^2 \sum_{x \in P^*} |A_x|^2 \leq 4L^2 (\tilde{\Delta})^{2-s} \mathbb{E}_s(A) \leq 4L^2 (16L)^{1-s/2} \mu_0(T_A^{A \circ A})^{1-s/2} \mathbb{E}_s(A).$$

Now using tensor trick, we obtain (106). Inequality (107) can be derived similarly. This completes the proof.  $\square$

Example from Remark 41 shows that all inequalities of the theorem above are sharp. We need in  $\mu_0(T_A^{A \circ A})$  in bound (106) as Example 18 asserts us.

In Theorem 43 the quantity  $\mu_0(T_A^{(A \circ A)^{k-1}})$  or, more precisely, quantity  $\mu_0(T_A^P)$  for some popular set  $P$  has appeared. The next lemma shows that one can easily estimate former eigenvalue for large subset of  $A$ .

**Lemma 44** *Let  $A \subseteq \mathbf{G}$  be a set. There is  $A' \subseteq A$ ,  $|A'| \geq |A|/2$  such that  $\mu_0(T_{A'}^P) \leq \frac{2\mathbb{E}(A)}{\Delta|A|}$  for any set  $P \subseteq \{x : |A_x| \leq \Delta\}$  and any real number  $\Delta > 0$ . In particular,  $\mu_0(T_{A'}^{A \circ A}) \leq \frac{2\mathbb{E}(A)}{|A|}$ .*

*Proof.* Let

$$A_1 = \{x : ((A * A) \circ A)(x) > 2\mathbb{E}(A)/|A|\}.$$

It is easy to see that  $|A_1| < |A|/2$ . Put  $A' = A \setminus A_1$  and let  $f$  be the main eigenfunction of the operator  $T_{A'}^P$ . Let also  $\mu_0 = \mu_0(T_{A'}^P)$ . We have

$$\mu_0 f(x) = A'(x)(P * f)(x).$$

Summing over  $x \in A'$  and using the definition of the set  $A'$ , we obtain

$$\mu_0 \sum_x f(x) = \sum_x f(x)(P \circ A')(x) \leq \Delta^{-1} \sum_x f(x)((A \circ A) \circ A)(x) =$$

$$= \Delta^{-1} \sum_x f(x) ((A * A) \circ A)(x) \leq \Delta^{-1} \frac{2E(A)}{|A|} \cdot \sum_x f(x)$$

and we are done.  $\square$

We need in an analog of a definition from [23].

**Definition 45** Let  $\alpha > 1$  be a real number,  $\beta, \gamma \in [0, 1]$ . A set  $A \subseteq \mathbf{G}$  is called  $(\alpha, \beta, \gamma)$ -connected if for any  $B \subseteq A$ ,  $|B| \geq \beta|A|$  the following holds

$$E_\alpha(B) \geq \gamma \left( \frac{|B|}{|A|} \right)^{2\alpha} E_\alpha(A).$$

Thus, a set from Theorem 2 is a  $(2, \beta, \gamma)$ -connected set with  $\beta, \gamma \gg 1$ . As was proved in [23] that for  $\alpha = 2$  every set  $A$  always contains large connected subset.

We obtain a consequence of Theorem 43 for connected sets  $A$ . Our inequality (113) below shows that there is a nontrivial connection between  $E(A)$  and  $E_s(A)$ ,  $1 \leq s \leq 2$  in the case.

**Corollary 46** Let  $A \subseteq \mathbf{G}$  be a set, and  $\beta, \gamma \in [0, 1]$ . Suppose that  $A$  is  $(2, \beta, \gamma)$ -connected with  $\beta \leq 1/2$ . Then there are two  $2^{-6}\gamma L^{-2}$ -popular dual sets  $P, P^*$  such that

$$\Delta \Delta^* \leq \frac{2^8 L^2 E(A)}{\gamma |A|}, \quad (110)$$

$$L^{-5} \gamma^3 2^{-21} |A|^2 \leq |P| |P^*|, \quad (111)$$

and

$$L^{-3} \gamma^2 2^{-13} E(A) |A| \leq \sigma_P(A) \sigma_{P^*}(A). \quad (112)$$

Further for any  $s \in [1, 2]$  the following holds

$$E_s(A) \geq 2^{-5} \gamma |A|^{1-s/2} E^{s/2}(A). \quad (113)$$

**Proof.** Let

$$P_j = \{x : 2^{j-1} \gamma E(A) / (2^5 |A|^2) < |A_x| \leq 2^j \gamma E(A) / (2^5 |A|^2)\}, \quad j \in [L].$$

Applying Lemma 44, we find a set  $A' \subseteq A$ ,  $|A'| \geq |A|/2$  such that for any  $j$  the following holds  $\mu_0(T_{A'}^{P_j}) \leq 2E(A)/(\Delta_j |A|)$ . Here  $\Delta_j = 2^j \gamma E(A) / (2^5 |A|^2)$ . By connectedness, we have

$$\gamma 2^{-5} E(A) \leq 2^{-1} E(A') \leq \sum_{j=1}^L \sum_{x \in P_j} (A' \circ A')^2(x)$$

and for some  $j \in [L]$  there is  $P = P_j$  such that

$$\gamma 2^{-5} L^{-1} E(A) \leq 2^{-1} L^{-1} E(A') \leq \sum_{x \in P} (A' \circ A')^2(x) = E_P(A'). \quad (114)$$

Of course

$$\mathbf{E}_P(A') \leq \mathbf{E}_P(A) \leq \Delta \sigma_P(A).$$

Consider  $P^*$  and put  $\Delta = \Delta_j$ . From (114) it follows that  $P, P^*$  are  $2^{-6}\gamma L^{-2}$ -popular dual sets. Applying the arguments of Theorem 43 for second estimate from (114), we obtain

$$2^{-2}L^{-2}\mathbf{E}(A') \leq \mu_0(\mathbf{T}_{A'}^P)\sigma_{P^*}(A') \leq \frac{2\mathbf{E}(A)}{\Delta|A|} \cdot \sigma_{P^*}(A). \quad (115)$$

Multiplying the last inequality by  $\Delta^*$  and using the connectedness again, we get

$$2^{-2}L^{-2}\Delta\Delta^*\mathbf{E}(A') \leq \frac{2\mathbf{E}(A)}{|A|}2\mathbf{E}(A) \leq 2^6\gamma^{-1}\mathbf{E}(A')\frac{\mathbf{E}(A)}{|A|}$$

and we obtain (110) for our  $P, P^*$ .

Further, multiplying estimate (115) by  $\sigma_P(A)$  and recalling (114), we have

$$\gamma 2^{-6}L^{-2}\mathbf{E}(A)\sigma_P(A)\Delta \leq \frac{2\mathbf{E}(A)}{|A|} \cdot \sigma_{P^*}(A)\sigma_P(A).$$

By the definition of the set  $P$  and inequality (114), we get

$$\gamma 2^{-6}L^{-2}\mathbf{E}(A) \cdot \gamma 2^{-6}L^{-1}\mathbf{E}(A) \leq \frac{2\mathbf{E}(A)}{|A|} \cdot \sigma_{P^*}(A)\sigma_P(A)$$

and we obtain (112). Combining (110) and (112), we get (111).

Finally, applying inequality (106) of Theorem 43, we have

$$2^{-4}\gamma\mathbf{E}(A) \leq \mathbf{E}(A') \leq \mu_0^{1-s/2}(\mathbf{T}_{A'}^{A' \circ A'})\mathbf{E}_s(A') \leq \mu_0^{1-s/2}(\mathbf{T}_{A'}^{A \circ A})\mathbf{E}_s(A) \leq \left(\frac{2\mathbf{E}(A)}{|A|}\right)^{1-s/2} \mathbf{E}_s(A)$$

and (113) follows.  $\square$

The example from Remark 41 shows that inequality (113) cannot be improved. In the situation  $P = P^* = \bigsqcup_{j=1}^k H_j$ , so it is natural to call the set  $A$  from the example as "self-dual" set. Such sets have some interesting properties, for example, by corollary above they always have relatively large  $\sigma_P(A)$  and small  $\Delta$ . The set from Remark 39, see also example 42, says that even for connected  $A$  estimate (113) does not satisfy for  $s > 2$ . So, in the region trivial estimates  $\mathbf{E}_{s_2}(A) \leq \mathbf{E}_{s_1}(A)|A|^{s_2-s_1}$ ,  $s_2 \geq s_1$  can be sharp. Finally, example 18 shows that inequality (113) cannot hold for any  $A$ , generally speaking, we need in connectedness of  $A$ , or, similarly, we need in  $\mu_0(\mathbf{T}_A^{A \circ A})$  not just  $\mathbf{E}(A)/|A|$  to use inequality (106) in the case of an arbitrary  $A$ .

Thus, in general, even for connected set  $A$  we cannot find  $P \subseteq A - A$  such that, roughly speaking,  $\mathbf{E}_P(A) \gg \mathbf{E}(A) = |A|^3/K$  with  $\sigma_P(A)$  greater than  $|A|^2/K^{1/2}$  and  $\Delta \ll |A|/K^{1/2}$ . Nevertheless if we know lower bounds for  $\mathbf{E}_s(A)$ ,  $s < 2$  then estimates of Corollary 46 can be improved. We show this in the next two statements.



**Proposition 47** *Let  $A \subseteq \mathbf{G}$  be a set,  $s \in (1, 2]$  and  $\beta, \gamma \in [0, 1]$ . Suppose that  $A$  is  $(s, \beta, \gamma)$ -connected with  $\beta \leq 1/2$ . Then there are two*

$$(2^{3s}\gamma^{-1}L^s)^{-1/(s-1)}\mathbf{E}_s^{1/(s-1)}(A)|A|^{-(4-2s)/(s-1)}\mathbf{E}^{-1}(A)$$

*-popular dual sets  $P, P^*$  such that*

$$\mathbf{E}_s(A)|A|^{s-1}\Delta(\Delta^*)^{s-1} \leq 2^{4s+3}\gamma^{-1}L^{s+1}\mathbf{E}^s(A), \quad (116)$$

*and*

$$\mathbf{E}_s^2(A)|A|^{s-1} \leq 2^{6s+1}\gamma^{-2}L^{s+1}\mathbf{E}^{s-1}(A)\sigma_P^{s-1}(A)\sigma_{P^*}^{3-s}(A). \quad (117)$$

**Proof.** Let

$$P_j = \{x : 2^{j-1}\gamma\mathbf{E}_s(A)/(2^{1+2s}|A|^2) < |A_x|^{s-1} \leq 2^j\gamma\mathbf{E}_s(A)/(2^{1+2s}|A|^2)\}, \quad j \in [L].$$

Applying Lemma 44, we find a set  $A' \subseteq A$ ,  $|A'| \geq |A|/2$  such that for any  $j$  the following holds  $\mu_0(\mathbf{T}_{A'}^{P_j}) \leq 2\mathbf{E}(A)/(\Delta_j|A|)$ . Here  $\Delta_j^{s-1} = 2^j\gamma\mathbf{E}_s(A)/(2^{1+2s}|A|^2)$ . By connectedness, we have

$$\gamma 2^{-1-2s}\mathbf{E}_s(A) \leq 2^{-1}\mathbf{E}_s(A') \leq \sum_{j=1}^L \sum_{x \in P_j} (A' \circ A')^s(x) \quad (118)$$

and for some  $j \in [L]$  there is  $P = P_j$  such that

$$\gamma 2^{-1-2s}L^{-1}\mathbf{E}_s(A) \leq 2^{-1}L^{-1}\mathbf{E}_s(A') \leq \sum_{x \in P} (A' \circ A')^s(x) = \mathbf{E}_s^P(A'). \quad (119)$$

Of course

$$\mathbf{E}_s^P(A') \leq \mathbf{E}_s^P(A) \leq \Delta^{s-1}\sigma_P(A).$$

By Hölder's inequality, we have

$$\mathbf{E}_s^P(A') \leq \mathbf{E}_P^{s-1}(A')\sigma_P^{2-s}(A'). \quad (120)$$

Now let  $P^*$  be a dual to the set  $P$ . In particular

$$\mathbf{E}_P(A') \leq 2L \sum_{\alpha} \mu_{\alpha}(\mathbf{T}_{A'}^P) \langle \mathbf{T}f_{\alpha}, f_{\alpha} \rangle \leq \frac{4L\mathbf{E}(A)}{\Delta|A|} \sigma_{P^*}(A), \quad (121)$$

where  $\{f_{\alpha}\}$  are the eigenfunctions of the operator  $\mathbf{T}_{A'}^P$  and  $\mathbf{T}$  is the operator defined by (101) with  $\mathcal{P} = P^*$ . From (119), (120) and a trivial upper bound  $\sigma_P(A) \leq |A|^2$  it follows that  $P$  and  $P^*$  are

$$(2^{3s}\gamma^{-1}L^s)^{-1/(s-1)}\mathbf{E}_s^{1/(s-1)}(A)|A|^{-(4-2s)/(s-1)}\mathbf{E}^{-1}(A)$$

-popular dual sets. Put  $\sigma = \sigma_P(A)$  and  $\sigma^* = \sigma_{P^*}(A)$ . Substitution (121) into (120) gives us in view of inequality (119) that

$$\mathbf{E}_s(A)\Delta^{s-1} \leq 2^{4s-1}\gamma^{-1}L^s \left( \frac{\mathbf{E}(A)}{|A|} \right)^{s-1} (\sigma^*)^{s-1} \sigma^{2-s}. \quad (122)$$

Multiplying by  $\sigma$ , and using (118), we obtain

$$\mathbf{E}_s^2(A) \leq 2^{6s+1} \gamma^{-2} L^{s+1} \left( \frac{\mathbf{E}(A)}{|A|} \right)^{s-1} (\sigma^*)^{s-1} \sigma^{3-s}$$

and (117) is proved.

Similarly, multiplying inequality (122) by  $(\Delta^*)^{s-1}(\Delta)^{2-s}$  and using estimates

$$\Delta \sigma \leq 2\mathbf{E}(A), \quad \Delta^* \sigma^* \leq 2\mathbf{E}(A),$$

we have

$$\mathbf{E}_s(A) |A|^{s-1} \Delta (\Delta^*)^{s-1} \leq 2^{4s+3} \gamma^{-1} L^{s+1} \mathbf{E}^s(A).$$

This completes the proof.  $\square$

For example, if  $\mathbf{E}(A) = |A|^3/K$ ,  $\mathbf{E}_{3/2}(A) = |A|^{5/2}/K^{1/2}$  then from (116) it follows that  $\min\{\Delta, \Delta^*\} \ll_L |A|/K^{2/3}$  instead of  $\min\{\Delta, \Delta^*\} \ll_L |A|/K^{1/2}$  which is a consequence of Theorem 43 provided by we have an appropriate bound for  $\mu_0(\mathbf{T}_A^{A \circ A})$  or Corollary 46.

**Corollary 48** *Let  $A \subseteq \mathbf{G}$  be a set,  $|A - A| \leq K|A|$ , and  $s \in (1, 2]$  be a real number. Then there are two*

$$(cL^s)^{1/(s-1)} \frac{|A|^3}{K\mathbf{E}(A)} \tag{123}$$

*–popular dual sets  $P, P^*$ , where  $c > 0$  is an absolute constant, such that*

$$\Delta (\Delta^*)^{s-1} \ll L^{s+1} \cdot \frac{K^{s-1} \mathbf{E}^s(A)}{|A|^{2s}}, \tag{124}$$

*and*

$$|A|^{3s+1} \ll K^{2(s-1)} L^{s+1} \mathbf{E}^{s-1}(A) \sigma_P^{s-1}(A) \sigma_{P^*}^{3-s}(A). \tag{125}$$

**Proof.** Let

$$P_j = \{x : 2^{j-1}|A|^{s-1}/(2^{2s+1}K^{s-1}) < |A_x|^{s-1} \leq 2^j|A|^{s-1}/(2^{2s+1}K^{s-1})\}, \quad j \in [L].$$

Applying Lemma 44, we find a set  $A' \subseteq A$ ,  $|A'| \geq |A|/2$  such that for any  $j$  the following holds  $\mu_0(\mathbf{T}_{A'}^{P_j}) \leq 2\mathbf{E}(A)/(\Delta_j|A|)$ . Here  $\Delta_j^{s-1} = 2^j|A|^{s-1}/(2^{2s+1}K^{s-1})$ . Application of Hölder inequality gives

$$\mathbf{E}_s(A') \geq 2^{-2s}|A|^{s+1}/K^{s-1}.$$

Then for some  $j \in [L]$  and  $P_j = P$  the following holds

$$\mathbf{E}_s(A') \leq 2L \sum_{x \in P} |A'_x|^s.$$

After that apply the arguments of Proposition 47. This concludes the proof.  $\square$

Now we try to prove an analog of Theorem 38 with a weaker assumption on the set  $A$ , namely, the largeness of the additive energy. More precisely, we obtain a lower bound for  $E_4(A)$ , and the existence of structural subset  $A' \subseteq A$  follows similarly as in Theorem 38. Something can be proved using dual technique, but we add one more optimization in the argument. Our result is a very simple, the reason why we cannot get bounds similar to Theorem 38 is discussed after Proposition 49.

**Proposition 49** *Let  $A \subseteq \mathbf{G}$  be a set,  $E(A) = |A|^3/K$ , and  $T_4(A) = M|A|^7/K^3$ . Then*

$$E_4(A) \geq \frac{|A|^5}{2^5 L^{10/3} M^{1/3} K^{7/3}}. \quad (126)$$

**Proof.** Let  $P$  be a popular difference set, such that

$$E(A) \leq 2L \sum_{x \in P} |A_x|^2. \quad (127)$$

From (127) we, clearly, have

$$E_4(A) \geq (8L)^{-1} K^{-1} \Delta^2 |A|^3. \quad (128)$$

On the other hand from the arguments of Theorem 38 it follows that

$$\left( \frac{E(A)}{2L|A|} \right)^8 \leq E_4(A) \Delta^4 T_4(A) \leq E_4(A) \Delta^4 \frac{M|A|^7}{K^3}. \quad (129)$$

Combining (128), (129) and making an optimization over  $\Delta$ , we obtain the result. This completes the proof.  $\square$

The example from the Remark 41 shows that  $K^{7/3}$  in (126) cannot be replaced by something smaller than  $K^2$ . The reason why we have exactly  $K^2$  in the example is clear. Indeed, it is easy to see that there are  $[K^{1/2}]$  eigenvalues equals, roughly,  $\mu_0(T_A^{A \circ A})$  in the case and our approximation of the sum  $\sum_{\alpha} \mu_{\alpha}^4(T_A^{A \circ A})$  by just zero term is inappropriate. Quick calculations show that using all  $[K^{1/2}]$  eigenvalues in the sum  $\sum_{\alpha} \mu_{\alpha}^4(T_A^{A \circ A})$ , we get  $K^2$ . It is interesting to obtain better estimate than (126).

The same example shows that the way of finding structured  $A' \subseteq A$  obtaining lower bounds for  $E_4(A)$  is inappropriate in general. Indeed, in the proof of Theorem 38 we use Lemma 10 which asserts that  $E_4(A) = \sum_{s,t} E(A_s, A_t)$ . But in the example for typical  $(s, t)$  the energy  $E(A_s, A_t)$  is pretty small and the average arguments give almost nothing.

**Remark 50** *Of course the assumption of Theorem 38 is stronger than the condition of Proposition 49. Such type of assumptions, namely, lower bounds for  $E_s(A)$ ,  $s \leq 2$  can be interpreted as closeness of  $A$  to be a set with small doubling. Indeed, by Hölder inequality all such conditions are included to each other and if  $A$  has small doubling then all  $E_s(A)$ ,  $s \geq 1$  are large.*

We conclude the section considering one more example of dual sets. Let

$$P = \{x : E(A, A_x) \geq |A_x|^2 E_3(A) (2E(A))^{-1}\}.$$

Then by Lemma 10

$$2^{-1} E_3(A) \leq \sum_{x \in P} E(A, A_x) = \sum_{x, y} A(x) A(y) (A \circ A)(x - y) \mathcal{C}_3(P, A, A)(x, y).$$

The last expression looks similar to (98). As above define  $P^*$  by formula

$$P^* = \{x : |A_x| \geq E_3(A) (4E(A))^{-1}\}.$$

Thus

$$2^{-2} E_3(A) \leq \sum_{x, y} A(x) A(y) (A \circ A)(x - y) P^*(x - y) \mathcal{C}_3(P, A, A)(x, y). \quad (130)$$

Applying Cauchy–Schwartz inequality to estimate (130), we get

$$2^{-4} E_3^2(A) \leq E_3^{P^*}(A) \cdot \sum_{x, y \in P} \mathcal{C}_3^2(A)(x, y).$$

In particular,

$$\sum_{x \in P^*} |A_x|^3 \gg E_3(A)$$

for a dual set  $P^*$ . Using (130) one can obtain an analog of Theorem 43 for such type of dual sets, of course.

## References

- [1] M. BATEMAN, N. KATZ, *New bounds on cap sets*, arXiv:1101.5851v1 [math.CA] 31 Jan 2011
- [2] M. BATEMAN, N. KATZ, *Structure in additively nonsmoothing sets*, arXiv:1104.2862v1 [math.CO] 14 Apr 2011.
- [3] J. BOURGAIN, *More on the sum-product phenomenon in prime fields and its applications*, Int. J. Number Theory 1:1 (2005), 1–32.
- [4] J. BOURGAIN, *Estimates on exponential sums related to the Diffie–Hellmann distributions*, to appear in GAFA.
- [5] A. CARBERY, *A multilinear generalization of the Cauchy–Schwarz inequality*, to appear in Proc. AMS.
- [6] A. CARBERY, *An automatic proof of a multilinear generalization of the Cauchy–Schwarz inequality?*, preprint.

- [7] A. A. GLIBICHUK, S. V. KONYAGIN, *Additive properties of product sets in fields of prime order*, Proceedings of Centre de Recherches Mathématiques, v. 43 (Montreal, 2006), American Math. Soc., Providence, 279–286.
- [8] W. T. GOWERS *A new proof of Szemerédi’s theorem for arithmetic progressions of length four*, Geom. func. anal. **8** (1998) 529–551.
- [9] W. T. GOWERS *A new proof of Szemerédi’s theorem*, Geom. func. anal. **11** (2001) 465–588.
- [10] T. G. F. JONES, O. ROCHE-NEWTON, *Improved bounds on the set  $A(A + 1)$* , arXiv:1205.3937v1 [math.CO].
- [11] R. HORN, C. JOHNSON, *Matrix Analysis*, Cambridge University Press, Cambridge, 1985, xiii+561 pp.
- [12] A. IOSEVICH, S. V. KONYAGIN, M. RUDNEV, V. TEN, *On combinatorial complexity of convex sequences*, Discrete Comput. Geom. **35** (2006), 143–158.
- [13] S. V. KONYAGIN, *Estimates for trigonometric sums and for Gaussian sums*, IV International conference ”Modern problems of number theory and its applications”. Part 3 (2002), 86–114.
- [14] S. V. KONYAGIN, M. RUDNEV, *On new sum–product type estimates*, preprint.
- [15] L. LI, *On a theorem of Schoen and Shkredov on sumsets of convex sets*, arXiv:1108.4382v1 [math.CO].
- [16] L. LI, O. ROCHE-NEWTON, *Convexity and a sum–product type estimate*, arXiv:1111.5159v1 [math.CO].
- [17] M. RUDNEV, *An improved sum–product inequality in fields of prime order*, arXiv:1011.2738v2 [math.CO].
- [18] T. SANDERS, *Approximate (abelian) groups*, arXiv:1212.0456v1 [math.CA] 3 Dec 2012.
- [19] T. SCHOEN, *New bounds in Balog–Szemerédi–Gowers theorem*, preprint.
- [20] T. SCHOEN, I. D. SHKREDOV, *Additive properties of multiplicative subgroups of  $\mathbb{F}_p$* , Quart. J. Math., **63**:3 (2012), 713–722.
- [21] T. SCHOEN, I. D. SHKREDOV, *On sumsets of convex sets*, Comb. Probab. Comput. **20** (2011), 793–798.
- [22] T. SCHOEN, I. D. SHKREDOV, *Higher moments of convolutions*, arXiv:1110.2986v3 [math.CO] 21 Sep 2012.
- [23] I. D. SHKREDOV, *On Sets with Small Doubling*, Mat. Zametki, **84**:6 (2008), 927–947.
- [24] I. D. SHKREDOV, *Some applications of W. Rudin’s inequality to problems of combinatorial number theory*, Uniform Distribution Theory, **6**:2 (2011), 95–116.

- [25] I. D. SHKREDOV, *Some new inequalities in additive combinatorics*, arXiv:1208.2344v2 [math.CO].
- [26] I. D. SHKREDOV, *On Heilbronn's exponential sum*, Quart. J. Math., doi 10.1093/qmath-has037.
- [27] I. D. SHKREDOV, I. V. V'UGIN, *On additive shifts of multiplicative subgroups*, Mat. Sbornik, 203:6 (2012), 81–100.
- [28] T. TAO, V. VU, *Additive combinatorics*, Cambridge University Press 2006.

Division of Algebra and Number Theory,

Steklov Mathematical Institute,

ul. Gubkina, 8, Moscow, Russia, 119991

and

Delone Laboratory of Discrete and Computational Geometry,

Yaroslavl State University,

Sovetskaya str. 14, Yaroslavl, Russia, 150000

and

IITP RAS,

Bolshoy Karetny per. 19, Moscow, Russia, 127994

`ilya.shkredov@gmail.com`